13.2 Integrals of vector valued function 1/28/20

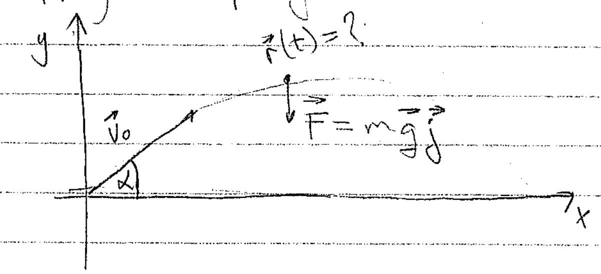
$\vec{r}(t) = x(t)i + y(t)j + z(t)k$

$$\int_{t=a}^{t=b} r(t)\,dt = \left(\int_a^b x(t)\,dt\right)i + \left(\int_a^b y(t)\,dt\right)j + \left(\int_a^b z(t)\,dt\right)k$$

Newton's Law of Motion

$$\vec{F} = m\vec{a} \Rightarrow \vec{a} = \vec{r}''(t)$$

Apply to projectile (ideal)



V_0 – intial velocity vector from intial position $\vec{r}_0 = \vec{0}$

$\vec{r}(0) = 0$

$\vec{r}'(0) = \vec{V}_0$

$\cancel{m}\vec{a} = -\cancel{m}g\vec{j}$

$\vec{r}''(t) = \vec{a} = -g\vec{j}$ ← constant vector

$$\vec{v}(t) = \int a(t)\,dt = \int -gj\,dt = -gtj + \vec{C}$$

$t=0 \quad V_0 = V(0) = \vec{C}$

$$v(t) = -gtj + \vec{V}_0$$

$$\vec{r}(t) = \int v(t)\,dt = \int(-gtj + V_0)\,dt = -g\frac{t^2}{2}\vec{j} + t\vec{V}_0 + \vec{\tilde{C}}$$

$\vec{0} = \vec{r}(0) = \tilde{C} \qquad \vec{r}(t) = t\vec{V}_0 - \frac{1}{2}gt^2\vec{j} =$

$= t|V_0|\cos\alpha\, i + t|V_0|\sin j - \frac{1}{2}gt^2 j$

$$\vec{r}(t) = \langle \underset{x}{\underset{\shortparallel}{t|V_0|\cos\alpha}},\ \underset{y}{\underset{\shortparallel}{t|V_0|\sin\alpha - \frac{1}{2}gt^2}} \rangle$$

$h_{max} = h(t_{max})$

$$R = \frac{|V_0|^2}{g}\sin(2\alpha) \qquad \text{max range if } 45^\circ$$

range horizontal

Ex exam

i) area of parallelogram by u and w
ii) angle u & v
iii) $P_u v$
iv) volume generated by u, v, w ?
v) eq. of a plane, contains origin, is parallel to u & w

i) $|\vec{u} \times \vec{w}| = \text{area}$ ii) $\cos\theta = \frac{u \cdot v}{|u| \cdot |v|}$

iii) $P_u v = \frac{v \cdot u}{|u|^2} u$ iv) $|(u \times v) \cdot w|$

v) $\vec{n} = u \times w$
$P_0(0,0,0)$

From exam

(6) $r(t) = {}^{3}/\sqrt{2}\cos t\, i + 3\sin t\, j + \frac{3}{\sqrt{2}}\cos t\, k$

a) Find parametric eq. of the tangent at $t = \frac{\pi}{4}$

$$r'(t) = -\frac{3}{\sqrt{2}}\sin t\, i + 3\cos t\, j - \frac{3}{\sqrt{2}}\sin t\, k$$

$$\vec{v} = < -\frac{3}{2}, \frac{3}{\sqrt{2}}, -\frac{3}{2} > \qquad P_0 = < \frac{3}{2}, \frac{3\sqrt{2}}{2}, \frac{3}{2} >$$

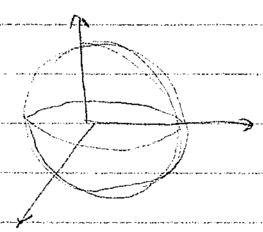
$$x = \frac{3}{2} - \frac{3}{2}t$$
$$y = \frac{3\sqrt{2}}{2} + \frac{3}{\sqrt{2}}t$$
$$z = 3/2 - 3/2\, t$$

b) Show that the curve lies on the sphere $x^2+y^2+z^2 = 9$ and on the certain plain

$$\left(\frac{3}{\sqrt{2}}\cos t\right)^2 + (3\sin t)^2 + \left(\frac{3}{\sqrt{2}}\cos t\right)^2 = \frac{9}{2}\cos^2 + 9\sin^2 + \frac{9}{2}\cos^2$$
$$= 9 \checkmark$$

$x = z \Rightarrow$ plane

$x - z = 0$



c) Radius $= 3$ since the plane $x - z = 0$ contains the origin which is center of sphere

13.3 length of a curve 02/04/2020

(1) $r(t) = \frac{3}{\sqrt{2}}\cos t\,\vec{i} + 3\sin t\,\vec{j} + \frac{3}{\sqrt{2}}\cos t\,\vec{k}$

find arc length $[0, 2\pi]$

$$S = \int_{t=t_0}^{t=t_1} |\vec{v}|\,dt$$

(speed)

$\vec{v} = \vec{r}'(t)$

$r'(t) = \langle -\frac{3}{\sqrt{2}}\sin t, 3\cos t, -\frac{3}{\sqrt{2}}\sin$

$$S = \int_0^{2\pi} 3\,dt = \boxed{6\pi}$$

$|\vec{v}| = \sqrt{\frac{9}{2}\sin^2 t + 9\cos^2 + \frac{9}{2}\sin^2 t}$

$|\vec{v}| = \sqrt{9} = 3$

$r(t)$

$t = 6\tilde{t}$ ← change of parameter

$\tilde{r}(\tilde{t}) = \vec{r}(6\tilde{t})$

graph of the new curve is the same as the old or
the speed is different (6 times bigger)

$$\tilde{\vec{v}} = \frac{d\tilde{\vec{r}}}{d\tilde{t}} = \frac{dr}{dt} \cdot \underbrace{\frac{dt}{d\tilde{t}}}_{6} = 6 \cdot \frac{dr}{dt} = 6\vec{v}$$

$r(t)$

change of parameter $t = \varphi(\tilde{t})$

$$\tilde{r}(\tilde{t}) = r(\varphi(\tilde{t})) = \langle x(\varphi(\tilde{t})), y(\varphi(\tilde{t})), z(\varphi(\tilde{t}))$$

$$\tilde{v} = \frac{d\tilde{r}}{d\tilde{t}} = \frac{dr}{dt} \cdot \frac{dt}{d\tilde{t}} = \vec{v} \cdot \varphi'(\tilde{t})$$

scalar

Arclength parameter s has the property $\left|\frac{d\vec{r}}{ds}\right| = $

$\vec{r}(t)$ where t is an arbitrary parameter

$$s = \int_{t_0}^{t} |\vec{r}'(t)|\,dt$$

$$\frac{ds}{dt} = \frac{d}{dt}\left(\int_{t_0}^{t} |\vec{r}\,'(t)|\,dt\right) = |\vec{r}\,'(t)| = \text{speed}(t)$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\vec{r}\,'(t)|}$$

$$s = \int_0^t |r'(t)|\,dt = \int_0^t 3\,dt = 3t \qquad \begin{array}{l} s = \varphi(t) \\ t = \varphi^{-1}(s) \end{array}$$

$$\vec{r}(s) = \frac{3}{\sqrt{2}}\cos\left(\frac{s}{3}\right)i + 3\sin\left(\frac{s}{3}\right)j + \frac{3}{\sqrt{2}}\cos\left(\frac{s}{3}\right)k$$

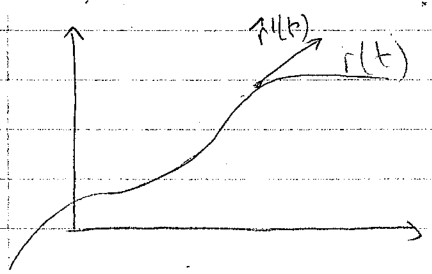
arc length param. of the initial curve

$$\left|\frac{dr}{ds}\right| = 1$$

13.4 Curvature

02/04/2020

$\vec{r}(t)$, $\vec{r}\,'(t)$ ← tangent vector at point t



$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

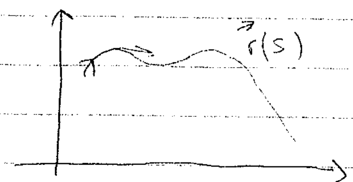
unit tangent vector

Def: A curve $r(t)$ is said to be smooth if is differentiable and $\vec{r}\,'(t) \neq 0$ for all t in its domain

If $\vec{r}(s)$ is in arclength parameterization $(|\vec{r}(s)| = 1)$

$$\vec{T}(s) = \vec{r}\,'(s)$$

a) Suppose $\vec{r}(s)$ is a smooth curve in arclength param.



$$\left|\frac{d\vec{T}}{ds}\right| \overset{\text{def}}{=} k(s)$$

$$\frac{ds}{dt} = |r'(t)| =$$

$$\frac{dt}{ds} = \frac{1}{|r'(t)|}$$

b) If $r(t)$ is a curve in an arbitrary param.

$$k(t) = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt} \cdot \frac{dt}{ds}\right| = \left|\frac{dT}{dt} \cdot \frac{1}{|V|}\right|$$

$$\boxed{k(t) = \frac{1}{|\vec{V}|} \cdot \left|\frac{d\vec{T}}{dt}\right| = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}'(t)|} \geq 0}$$

$$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$$

$x = a\cos t$

$y = a\sin t$

$$\vec{r}'(t) = \langle -a\sin, a\cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{a^2} = a$$

$$\vec{T}(t) = \frac{r'(t)}{|r'(t)|} = \langle -\sin t, \cos t \rangle$$

$$\vec{T}'(t) = \frac{d\vec{T}}{dt} = \langle -\cos t, -\sin t \rangle$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$k(t) = \frac{1}{a}$$

$r(t) = 3\cos t\, i + 2\sin t\, j$ $\quad \begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$ ellipse

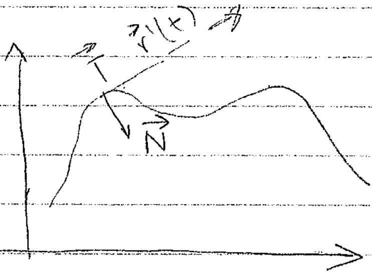
$r'(t) = \langle -3\sin t, 2\cos t \rangle$ $\quad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$|\vec{r}'(t)| = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{4 + 5\sin^2 t}$

$T(t) = \dfrac{\langle -3\sin t, 2\cos t \rangle}{\sqrt{4 + 5\sin^2 t}}$ too hard to solve

$T(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

13.4/13.5



- unit NORMAL vector to the curve

$|\vec{T}(t)|^2 = 1$

$\vec{T}(t) \cdot \vec{T}(t) = 1 \quad |'$

$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$

$T'(t) \cdot T(t) = 0$

$T'(t) \perp T(t)$

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} \Rightarrow \dfrac{dT}{dt} = \left|\dfrac{dT}{dt}\right| \cdot N$

perpendicular to $T(t)$ and pointing towards the concavity of the curve

$\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$ Frenet frame

unit by normal

$\dfrac{dT}{dt} = \left|\dfrac{dT}{dt}\right| N =$

$= K(t)\,|r'(t)| \cdot N$

combine

$\vec{v} = \vec{r}'(t) = |\vec{r}'(t)| \cdot \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = |\vec{r}'(t)| \cdot \vec{T}(t)$

$\vec{a} = \vec{r}''(t) = \dfrac{d}{dt}\left(|\vec{r}'(t)|\right) \cdot \vec{T}(t) + |\vec{r}'(t)| \cdot \dfrac{d\vec{T}}{dt}$

$\vec{a} = r''(t) = \dfrac{d}{dt}\left(|\vec{r}'(t)|\right) \cdot \vec{T}(t) + K(t)\,|r'(t)|^2\, \vec{N}$

$r'(t) \times r''(t) = \left[|r'(t)| \cdot T(t)\right] \times \left[\dfrac{d}{dt}\left(|r'(t)|\right) T + K\,|r'(t)|^2 N\right]$

$r'(t) \times r''(t) = K(t) \cdot |r'(t)|^3\, B$

$$r'(t) \times r''(t) = \kappa(t)\, |r'(t)|^3 \vec{B} \qquad |\vec{B}| = 1$$

$$|r'(t) \times r''(t)| = \kappa(t)\, |r'(t)|^3$$

$$\boxed{\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}}$$

$$\vec{B} = \frac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t) \times \vec{r}''(t)|}$$

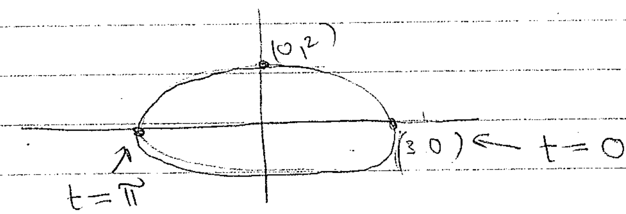
$$\vec{r}(t) = 3\cos t\, i + 2\sin t\, j$$

$$\vec{v} = \langle -3\sin t,\ 2\cos t \rangle$$

$$\vec{a} = \langle -3\cos t,\ -2\sin t \rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -3\sin t & 2\cos t & 0 \\ -3\cos t & -2\sin t & 0 \end{vmatrix} = (6\sin^2 t + 6\cos^2 t)\, k = 6\vec{k}$$

$$\kappa(t) = \frac{6}{\left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3}$$



$$\kappa(t) = \frac{6}{\left(\sqrt{4 + 5\sin^2 t}\right)^3}$$

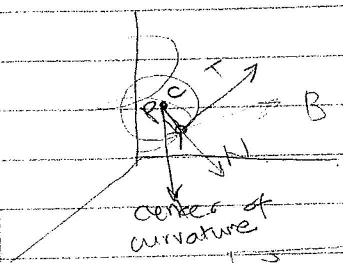
$$\kappa = \frac{6}{(\sqrt{4})^3} = \frac{6}{8}$$ ← curvature the biggest when $t = 0$, $t = \pi$

smallest $\frac{6}{(\sqrt{9})^3} = \frac{6}{27}$ when $t = \pi/2$, $3\pi/2$ for $\sin(t)$

$$\boxed{\vec{B} = \vec{k}}$$

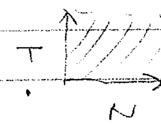
2/6/2020

Review : Curves in R^3 (13.4 / 13.5)



$\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ unit tangent

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ unit normal



$\boxed{\vec{B} = \vec{T} \times \vec{N}}$ unit binormal

osculating plane of the curve at point P is the plane generated by $\vec{T}$ and $\vec{N}$

$\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

For a circle of radius a the curvature is const. and $\kappa = 1/a$ constant for circle

$\rho(t) = \dfrac{1}{\kappa(t)}$

radius of curvature

Ex. Consider the $\vec{r}(t) = 3\cos t\, i + 3\sin t\, j + tk$

a) Compute the curvature

b) find the oscullating plane to the curve at the point P_0 corresponding to $t = \pi/4$

c) Find the radius of curvature at P and location of the center of curvature to P

a) $\vec{v} = \langle -3\sin t, 3\cos t, 1 \rangle$

$\vec{a} = \langle -3\cos t, -3\sin t, 0 \rangle$

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 1 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} = 3\sin t\, i - 3\cos t\, j + 9k$$

$$\vec{k}(t) = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{3\sqrt{10}}{(\sqrt{10})^3} \qquad |\vec{v} \times \vec{a}| = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10} \qquad |\vec{v}| = \sqrt{10}$$

$$\vec{k}(t) = \frac{3}{10} \rightarrow \text{constant curvature}$$

b) $$B = \frac{r'(t) \times r''(t)}{|r'(t) \times r''(t)|} = \frac{v \times a}{|v \times a|}$$

Point $P_0 \Leftrightarrow r(\pi/4) =$

$P_0 = \left\langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, \frac{\pi}{4} \right\rangle$

* normal to the osculating plane is

$\Rightarrow r'(\pi/4) \times r''(\pi/4) = \left\langle \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, 9 \right\rangle$

plane at P: $\frac{3\sqrt{2}}{2}\left(x - \frac{3\sqrt{2}}{2}\right) - \frac{3\sqrt{2}}{2}\left(y - \frac{3\sqrt{2}}{2}\right) + 9\left(z - \frac{\pi}{4}\right) = 0$

* radius of K at P:

$$\rho = \frac{1}{k} = \frac{10}{3}$$

* Center of curvature:

$$\vec{T} = \frac{1}{\sqrt{10}} \langle -3\sin t, 3\cos t, 1 \rangle$$

$$N = \frac{T'}{|T'|} = \frac{\cancel{\sqrt{10}}}{3} \cdot \frac{1}{\cancel{\sqrt{10}}} = \frac{1}{3} \langle -3\cos t, -3\sin t, 0 \rangle$$

$\vec{N} = \langle -\cos t, -\sin t, 0 \rangle$

$N(\pi/4) = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right\rangle$

Line that goes through P_0 in direction of $\vec{v}$.

$\vec{r} = r_0 + t\vec{N}(\pi/4)$ $\quad |N(\pi/4)| = 1$

center of curvature $\boxed{C = \vec{r_0} + \rho \cdot \vec{N}(\pi/4)}$

$$r(t) =$$

$$|r'(t)| = \frac{ds}{dt} \text{ speed}$$

$$v(t) = r'(t) = |r'(t)| \cdot T$$

$$a(t) = r''(t) = \frac{d}{dt}\left(|r'(t)|\right) \cdot \vec{T} + k(t) \cdot |r'(t)| \cdot N$$

$$a(t) = \underbrace{\frac{d^2 s}{dt^2}}_{a_T \text{ tangental}} T + k(t)\left(\frac{ds}{dt}\right)^2 \cdot \vec{N}$$

14.1 Multivariable

02/06/2020

$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \,/\, x_1, \dots, x_n \text{ are real numbers}\}$

Let D be a certain set in $\mathbb{R}^n$

Def: A real valued function f w/ domain D is a rule that assigns to any n-taple $(x_1, x_2, \dots x_n)$ from D exactly one value $w = f(x_1, x_2, \dots x_n)$

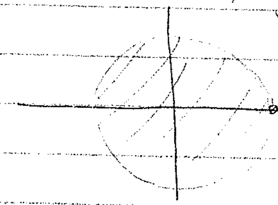
$f(x,y) = z = \sqrt{25 - x^2 - y^2}$ ← Domain $D = \{(x,y) \in \mathbb{R}^2 \mid 25 - x^2 - y^2 \geq 0\}$

$z = f(x,y) = \ln(y - x^2)$

$w = f(x,y,z) = x^2 + y^2 + z^2$

1

$x^2 + y^2 \leq 25$

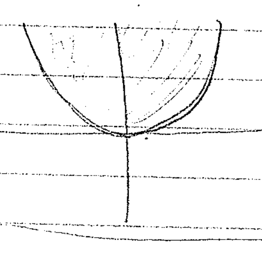


$(5,0)$ interior of plane with boundaries

$D = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 > 0\}$

$y > x^2$

2

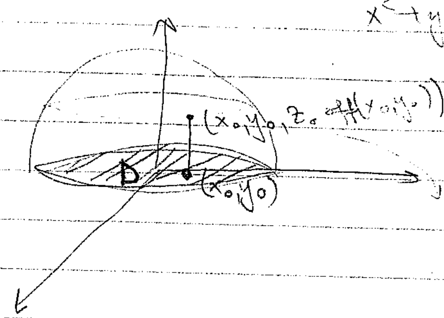


above the parabola without the boundaries

$D = \mathbb{R}^3$

3

$z = \sqrt{25 - x^2 - y^2}$ ← graph will be the upper-hemisphere of $x^2+y^2+z^2=25$



$f(5,0) = \sqrt{25-5^2-0} = 0$

$z_0 = f(2,3) = \sqrt{12} > 0$

$z = \ln(y - x^2)$ ← level curves $z = k$

$z = 0 \Rightarrow 0 = \ln(y-x^2)$

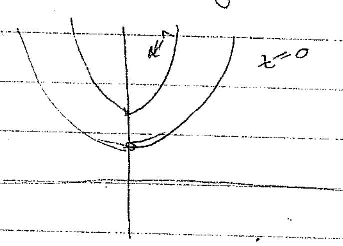
$y - x^2 = 1$

$y = 1 + x^2$

$z = k \quad k = \ln(y - x^2)$

$e^k = y - x^2$

$y = x^2 + e^k$



Ex. a) Find the domain of $z = f(x,y) = x^2 - y^2$

domain $D = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

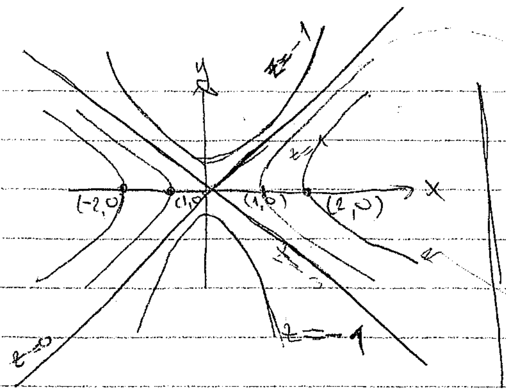
b) Describe the graph: hyperbolic paraboloid ("saddle

c) Draw the level curves of the function $z = x^2 - y^2$

$z = k \to$ level curve $k = x^2 - y^2$

$k = 0 \quad 0 = x^2 - y^2 \Leftrightarrow (x-y)(x+y) = 0 \quad x - y = 0 \quad x + y$

→ 0-level curve is a pair of lines



$k=1 \quad x^2-y^2=1$ hyperbola

$k=4 \quad x^2-y^2=4$

$k=-1 \quad x^2-y^2=-1$

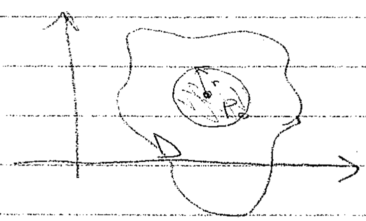
$-x^2+y^2=1$

14.1 — 2/11/2020

a) Suppose D is a set in $\mathbb{R}^2$ ($\mathbb{R}^3$)

Def: We say that a point $P_0 \in D$ is a point in interior of D if there exists $r>0$ so that $B(P_0,r) \subseteq D$

ball centered at P_0 w/ radius r



b) We say that P_0 is in the boundary of D (denote $P_0 \in \partial D$) if for any $r>0$, $B(P_0,r)$ will contain both points inside D and points outside D

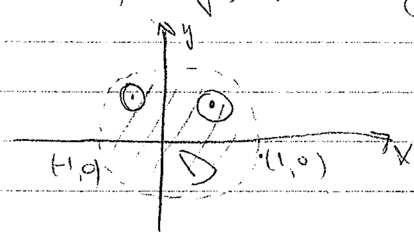
notation for boundary of D

$(B(P_0,r)\cap D \neq \emptyset)$ and $(B(P_0,r)\cap(\mathbb{R}^2 - D))$

Def: A set D in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is said to be open if every point of D is also an interior point of D

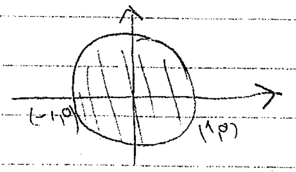
A set D — " — is said to be closed if $\partial D \subseteq D$

Ex.: $D=\{(x,y) \mid x^2+y^2<1\}$

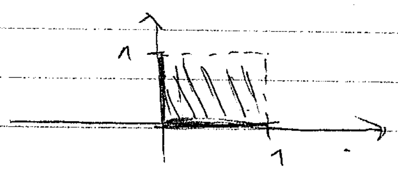


This set is open

(2) $D = \{(x,y) \mid x^2+y^2 \leq 1\}$ closed



(3) $D = \{(x,y) \mid 0 \leq x < 1, 0 \leq y < 1\}$



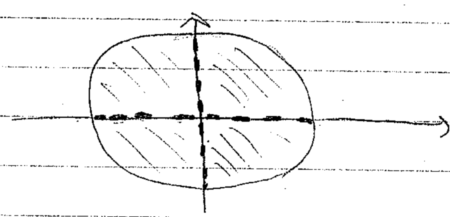
$\boxed{\text{neither open nor closed}}$

Ex Decribe the Domain of $f(x,y) = \dfrac{\sqrt{4-x^2-y^2}}{xy}$

a) and open or closed or neither

$D = \{(x,y) \mid x \neq 0, y \neq 0, 4-x^2-y^2 \geq 0\}$

$x^2+y^2 \leq 4$ closed disc



points on x and y-axis are no included

the set is neither open nor closed

$\partial D = \{(x,y) \mid x^2+y^2 = 4, (0,y)$ with $-2 \leq y \leq 2, (x,0)$ with $-2 \leq x \leq 2$

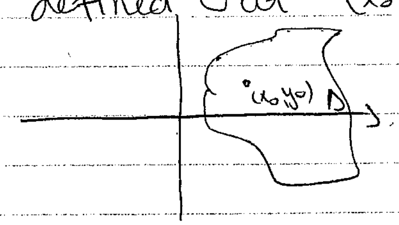
b) $\dfrac{1}{xy\sqrt{4-x^2-y^2}}$ open set

14.2

Limits for Functions of 2 variables $z = f(x,y)$

2/11/2020

Suppose $z = f(x,y)$ is define on some open set D containing the point (x_0, y_0), but possibly f is not defined at (x_0, y_0)



We say that the limit $\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}$ exists and is equal to L $(\mathbb{R})$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that for every point $(x,y) \in B((x_0,y_0), \delta)$ except (x_0,y_0)

$$|f(x,y) - L| < \varepsilon \qquad -\varepsilon < f(x,y) - L < \varepsilon$$

$$L - \varepsilon < f(x,y) < L + \varepsilon$$

Based on ε-δ definition of the limit one poses the basic properties of limits

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0 \qquad \lim_{(x,y)\to(x_0,y_0)} y = y_0$$

Suppose $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$ $(M, L \in \mathbb{R})$

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M$$

$$-\|- \quad (f(\ ) \cdot g(\ )) = L \cdot M$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad \boxed{\text{if } M \neq 0}$$

We say that $f(x,y)$ is continuous at (x_0,y_0) if $f(x_0,y_0)$ is defined

$$\lim_{(x,y)\to(x_0,y_0)} \frac{xy^2}{x^2+y^2} = 0$$

L

claim

$x^2+y^2 \geq y^2$

$$|f(x,y) - L| = \left|\frac{xy^2}{x^2+y^2} - 0\right| = \frac{|x|\cdot y^2}{x^2+y^2} \leq \frac{|x|\,\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = |x|$$

$$B((0,0),\alpha) = \{(x,y) \mid d((x,y),(0,0)) < \alpha\}$$
$$= \{(x,y) \mid x^2+y^2 < \alpha^2\}$$

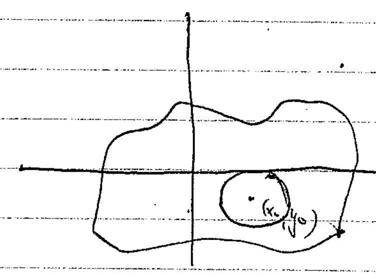
$$x^2 \leq x^2+y^2 \leq \alpha^2 \Rightarrow |x| < \alpha$$

def. Given $\varepsilon > 0$, choose $\alpha = \varepsilon$ and the definition is satisfied

* Consequences from the definition

Suppose $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$,

then $\lim_{\substack{(x(t),y(t))\to(x_0,y_0) \\ \text{as } t\to t_0}} f(x(t),y(t)) = L$



For any path $(x(t),y(t))$ approaching (x_0,y_0)

If there exist two paths $(x(t),y(t)) \to (x_0,y_0)$

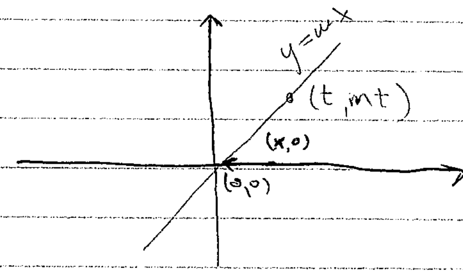
$(x_2(t),y_2(t)) \to (x_0,y_0)$

but $\Rightarrow$

$$\left.\begin{array}{l} \lim\limits_{t \to t_0} f(x(t), y(t)) = L_1 \\ \qquad\qquad\qquad\qquad\qquad \neq \\ \lim\limits_{t \to 0} f(x_2(t), y_2(t)) = L_2 \end{array}\right\} \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \text{ D.N.E.}$$

Claim

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{xy}{x^2+y^2}}_{f(x,y)} \quad \text{D.N.E.}$$



Path 1: approach $(0,0)$ along the x-axis

$(t,0)$ as $t \to 0$

$$\lim_{t\to 0} f(t,0) = \lim_{t \to 0} \frac{t \cdot 0}{t^2 + 0^2} = 0$$

Path 2: approach $(0,0)$ along y-axis

$$\lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{0 \cdot t}{0 + t^2} = 0$$

Path 3_m: $$\lim_{t\to 0} f(t, mt) = \lim_{t\to 0} \frac{t \cdot tm}{t^2 + (tm)^2} = \lim_{t \to 0} \frac{m\cancel{t^2}}{\cancel{t^2}(1+m^2)} =$$

approach $(0,0)$ on a line $y = mx$

$$= \frac{m}{1+m^2}$$

for different values of m we get different limits