Green's Theorm

11/30/21

Suppose R is a region in the x-y plane which is bounded, connected and simply connected (no holes inside)

Assume also that the boundry of R is a simple closed curve piece-wise smooth oriented counter clock wise

Let $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ be a vector field on a larger set containing R.

then

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \oint f(x,y)\,dx + g(x,y)\,dy \overset{\text{Green}}{=} \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{(\text{Curl}\,\vec{F})\cdot\vec{k}} \partial A}$$

Note :

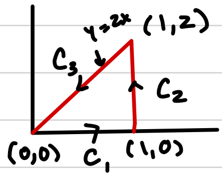
If $\vec{F}$ is a conservative vector field

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

both integrals in Green's Theorm will be zero example:

$$\oint x^2 y\,dx + x\,dy \qquad (f = x^2 y,\ g = x)$$

If c is the curve in the picture



Evaluate

Greens Theorm

$$\iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA = \int_0^1 \int_{y=0}^{y=2x} (1 - x^2)\,dy\,dx$$

$$\int_0^1 y - x^2 y \Big]_0^{2x} dx$$

$$\int_0^1 2x(1-x^2)\,dx$$

$$\int_0^1 2x - 2x^3\,dx$$

$$x^2 - \frac{1}{2}x^4 \Big]_0^1 = \frac{1}{2}$$

Using definition

$$\oint (\ldots) = \oint_{C1} (\quad) + \oint_{C2} (\quad) + \oint_{C3} (\quad)$$

$$0 + 2 + \left(-\frac{3}{2}\right) = \frac{1}{2}$$

where t goes from 1 to 0

C1: $x = t,\ y = 0$

$0 \le t \le 1$

$$\int_{C1} x^2 \cancel{y}^{\,0}\,dx + x\,\cancel{dy}^{\,0} = 0$$

C2: $x = 1,\ y = t$

$0 \le t \le 2$

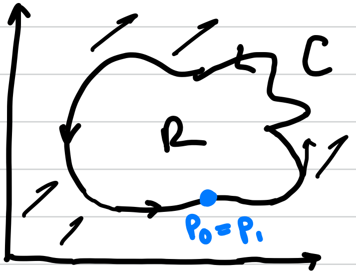
$$\int_{C2\ 0}^{2} x^2 y\,\cancel{dx} + x\,dy$$

$$1\,dt \Big]_0^2 = 2$$

C3: $x = t,\ y = 2t$

$$-\int_0^1 \left(t^2 \cdot 2t + t \cdot 2\right) dt$$

$$-\frac{3}{2}$$



$$\vec{F} = f(x,y)\hat{i} + g(x,y)\hat{j}$$

$$\oint \vec{F}\cdot\vec{T}\,ds = \oint \vec{F}\cdot\vec{r}'(t)\,dt$$

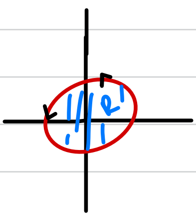
Applying Greens Theorm to compute Areas

$$\text{Area}(R) = \iint_R 1\,dA \overset{\text{Greens Theorms}}{=} \oint_C x\,dy = \int -y\,dx = \frac{1}{2}\oint -y\,dx + x\,dy$$

$$\frac{1}{2}\iint_R (1-(-1))\,dA = \iint 1\,dA$$

Example: Use Green's Theorm to find the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{x}{a} = \cos t \qquad \frac{y}{b} = \sin$$



$$\text{Area}(R) = \frac{1}{2}\oint (-y\,dx + x\,dy)$$

C: $x = a\cos t \quad y = b\sin t$

$t \in (0, 2\pi)$

$dx = -a\sin t\,dt \qquad dy = b\cos t$.

$$\text{Area}(R) = \iint_R 1\,dA$$

$$\frac{1}{2}\int_0^{2\pi} ab\sin^2 t\,dt + ab\cos^2 t\,dt$$

$$ab(\sin^2 t + \cos^2 t)$$

$$\frac{ab}{2}\int_0^{2\pi} 1\,dt$$

$$\frac{ab}{2}\cdot 2\pi = ab\pi$$

Greens Theorm for the Case of a region R with holes (R is not simply connected)

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$\oint_{C_{outer}} f(x,y)\,dx + g(x,y)\,dy - \oint_{C_2} (\quad) - \oint_{C_3} =$$

$$\iint_R f(x,y)\,dx + g(x,y)\,dy$$

Parametrized surfaces in $\mathbb{R}^3$

$\sigma = \vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

where (u,v) are parameters

Suppose the surface is a graph of a function $z = f(x,y)$

How can this be seen as a parametric surface?

$\langle x=u \,,\, y=v \,,\, z=f(u,v)\rangle = \vec{r}(u,v)$

$\vec{r}(u,v) = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$

σ: The sphere with center at $(0,0,0)$ and radius 3

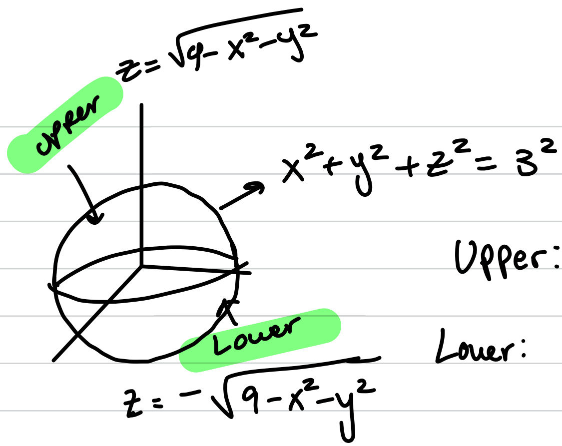
$x^2+y^2+z^2 = 3^2$

Can this be seen in a parametric way

$x = 3\sin\varphi\cos\theta \qquad y = 3\sin\varphi\cos\theta \qquad z = 3\cos\varphi$

$\vec{r}(\varphi,\theta) = (3\sin\varphi\cos\theta)\hat{i} + (3\sin\varphi\cos\theta)\hat{j} + (3\cos\varphi)\hat{k}$

$\varphi \in [0,\pi] \qquad \theta \in [0, 2\pi]$



Upper: $x = u$, $y = v$, $z = \sqrt{9 - u^2 - v^2}$

Lower: $x = u$

$$- \iint_T f(x,y,z) \cdot \underbrace{dS}_{\text{Element of Surface Area}}$$

$$dS = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

"

$$\iint_R f\big(x(u,v), y(u,v), z(u,v)\big) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

$R \rightarrow$ In the $u - v$ plane.

Final exam 12/7/21, Tuesday 9:30-11:50 12/2/21

- Comprehensive
  - review all previous exams, quizzes, worksheets

Concepts to Review

- basics on vector
  - dot, cross
- Lines & plane
- sphere, cylinders, quadric surfaces

Chpt 13

- Basics on curves 2D/3D

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Unit tangent $T = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$\vec{v}(t)$, $\vec{a}(t)$, speed, arc length

Chpt 14

- Partial derivatives
  - Chain rule
  - Gradient, directional derivatives
  - Tangent plane
  - local linear approximation
  - Critical point, optimization, Lagrange multipliers

Chpt 15

- Double/triple integrals cartesian coordinates
  - polar, cylindrical, spherical
  - change of variable, Jacobian
  - Mass, center of mass

Chpt 16

- Vector fields, divergence, curl
  - Line integrals, work
  - flux, circulation
  - conservative vector fields
  - Greens Theorem
  - surface integrals
  - Flux divergence Theorem

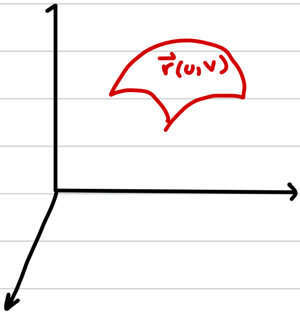
Chpt 17

(3 questions roughly)

Surface integrals, Flux Divergence Theorm

Let σ be a parametrized Surface

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ with u, v parameters varing in a certain region D in the uv plane



$$\iint_{\sigma} f(x,y,z)\, ds = \boxed{\iint_{D} f(x(u,v), y(u,v), z(u,v)) \cdot \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv}$$

element of Surface Area

$$ds = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\, dv$$

$$M = \iint \delta(x,y,z)\, ds$$

mass of Lamina

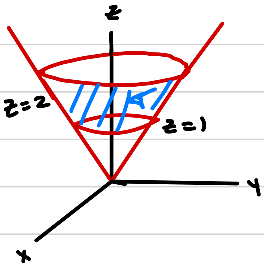
$$S = \iint_{\sigma} 1\, ds$$

Surface Area

Example #1

The Surface σ is the portion from the cone $z = \sqrt{x^2+y^2}$ cut btwn the planes $z=1$ and $z=2$

If the density at each point is $\delta(x,y,z) = z^2$, find the total mass of σ



$$M = \iint_{\Gamma} \delta(x,y,z)\, dS = \iint_{\Gamma} z^2\, dS$$

$\Gamma: z = f(x,y)$ $x = u$
$y = v$

$$\vec{r}(u,v) = \langle x=u,\ y=v,\ z=f(u,v) \rangle$$

$$\frac{\partial \vec{r}}{\partial u} = \left\langle 1,\ 0,\ \frac{\partial f}{\partial u} \right\rangle$$

cross product

$$\frac{\partial \vec{r}}{\partial v} = \left\langle 0,\ 1,\ \frac{\partial f}{\partial v} \right\rangle$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix} = \left(0 - \frac{\partial f}{\partial u}\right)\hat{i} - \left(\frac{\partial f}{\partial v} - 0\right)\hat{j} + (1-0)\hat{k}$$

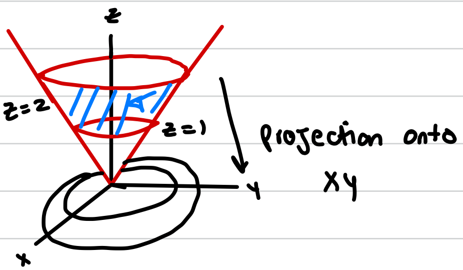
$$= -\frac{\partial f}{\partial u}\hat{i} - \frac{\partial f}{\partial v} j + \hat{k}$$

$$\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \sqrt{\left(-\frac{\partial f}{\partial u}\right)^2 + \left(-\frac{\partial f}{\partial v}\right)^2 + (1)^2}\right.$$

$$M = \iint \delta(u,v,f(u,v)) \cdot \left(\sqrt{\quad}\right) du\, dv = \iint \delta(x,y,f(x,y)) \cdot \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot dx\,dy$$

$$M = \iint_{\Delta} \left(\sqrt{x^2+y^2}\right)^2 \cdot \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dx\, dy$$

region in xy plane



$$z = f(x,y) = \sqrt{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

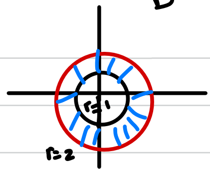
$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$ds = \sqrt{\underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{1} + 1}\;\; dx\, du$$

$$ds = \sqrt{2}\; dx\, dy$$

$$M = \iint_D (x^2+y^2)\cdot\sqrt{2}\;\partial A$$

D



$$\int_0^{2\pi}\int_1^2 (r^2)(\sqrt{2}\,\cdot)\,r\,dr\,\partial\phi$$

$$\sqrt{2}\int_0^{2\pi}\int_1^2 r^3\,dr\,\partial\phi$$

$$\sqrt{2}\int_0^{2\pi} \frac{r^4}{4}\Big]_1^2\,\partial\phi$$

$$\frac{(2)^4}{4}-\frac{(1)^4}{4}$$

$$\sqrt{2}\int_0^{2\pi}\frac{15}{4}$$

$$\boxed{\frac{30\pi\sqrt{2}}{4}=M}$$

Different Way to Solve with a different Parametrization

$z = \sqrt{x^2+y^2}$ ← with Cylindrical Coordinates

$x = r\cos\phi$

$y = r\sin\phi$

$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

$$\langle \overset{x}{r\cos\phi}, \overset{y}{r\sin\phi}, \overset{z}{r} \rangle = \vec{r}(r,\phi)$$

$$dS = \left\| \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \phi} \right\|$$

$$\frac{\partial \vec{r}}{\partial r} = \langle \cos\phi, \sin\phi, 1 \rangle \qquad \frac{\partial r}{\partial \phi} = \langle -r\sin\phi, r\cos\phi, 0 \rangle$$

$$\begin{vmatrix} i & j & k \\ \cos\phi & \sin\phi & 1 \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix} = (0 - r\cos\phi)\hat{\imath} - (0 + r\sin\phi)\hat{\jmath} + (r\cos^2\phi + r\sin^2\phi)\hat{k}$$

$$-r\cos\phi\,\hat{\imath} - r\sin\phi\,\hat{\jmath} + r\hat{k}$$

$$\langle -r\cos\phi, -r\sin\phi, r \rangle$$

$$dS = \sqrt{(r\cos\phi)^2 + (r\sin\phi)^2 + (r)^2}\; dr\,d\phi$$

$$\sqrt{r^2(1) + r^2}$$

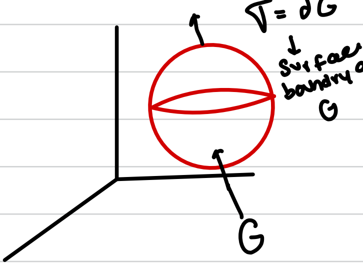
$$\sqrt{2r^2}\; dr\,d\phi = \underline{\sqrt{2}\, r\, dr\,d\phi = dS}$$

$$M=\int_0^{2\pi}\int_1^2 r^2\sqrt{z}\cdot r \, dr\, d\phi$$

⋮

Same result as other way.

Flux Divergence Theorm:



Let G be a bounded simply connected solid in 3D whose boundry is a surface σ oriented with an outward Normal.

Let $\vec{F}(x,y,z)=f(x,y,z)\hat{i}+g(x,y,z)\hat{j}+h(x,y,z)\hat{k}$ be a vector field defined on a set in 3D containing G.

Outward Flux of $\vec{F}$ through σ $=\iint_\sigma \vec{F}\cdot\vec{n}\, ds$

$\vec{n}$ ← unit vector Normal of σ

$$\vec{n}=\frac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\left\|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right\|}$$

$$=\iint_\sigma \vec{F}\cdot\frac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\cancel{\left\|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right\|}}\cdot\cancel{\left\|\frac{\partial \vec{r}}{\partial v}\times\frac{\partial \vec{r}}{\partial u}\right\|}\,du\,dx$$

$$=\iint_D \vec{F}\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)du\,dv$$

Flux-divergence Theorm

$$\text{Outward Flux} = \iint_{\sigma} \vec{F}\cdot\vec{n}\; dS = \underbrace{\iint_{D} (\operatorname{div}\vec{F})\, dV}_{\text{very often its easier to compute}}$$

$$\vec{F} = \frac{c\cdot\vec{r}}{\|\vec{r}\|^3}$$ → inverse square vector field → Not defined at the origin

$$\vec{F} = c\left\langle \frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3},\ \frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3},\ \frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3} \right\rangle$$

$$\operatorname{Div}\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$\underbrace{\frac{d}{dx}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{d}{dy}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{d}{dz}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)}_{=0}$$

If $(0,0,0) \notin G$, then Flux Theorm applies directly for $\vec{F} = \dfrac{c\vec{r}}{\|\vec{r}\|^2}$

∴ outward flux will be $= 0$

If the surface does contain the origin make a hole around the origin

$$\underset{\text{G-sphere}}{\iiint} \operatorname{div}\vec{F} = \underset{T}{\iint} \vec{F}\cdot\vec{n}\,ds + \underset{\text{Inward orientation}}{\iint} \vec{F}\cdot\vec{n}\,ds$$

$$\underset{T}{\iint} \vec{F}\cdot\vec{n}\,ds = \boxed{\underset{S}{\iint} F\cdot\vec{n}\,ds}$$

S → Outward Flux

→ Sphere of Normal radius

$$= \underset{S}{\iint} \frac{C\vec{r}}{\|\vec{r}\|^3}\cdot\frac{\vec{r}}{\|r\|} = \underset{S}{\iint} \frac{C\cdot|\vec{r}|^2}{\|\vec{r}\|^4} = \iint \frac{C}{\|\vec{r}^2\|}$$

$$\iint \frac{C}{r^2}\,ds =$$

Flux $= 4\pi C$