Exam #1 Thursday September 23 9/14/21

Covers Chpt 13 & 14

* Go to his website *

- previously taught courses
- go to calc #3

* wolframalpha.com for graphs

- Vector valued functions

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad t \in \mathbb{R}$$

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$\vec{r}\,'(t)$ will be a tangent to the $\vec{r}(t)$ curve

Definition: A curve $\vec{r}(t)$ is set to be differentiable if the components are all differentiable ($x'(t), y'(t), z'(t)$ all exist)

- A curve $\vec{r}(t)$ is said to be smooth if its differentiable and $\vec{r}\,'(t) \neq \vec{0}$

Example: $\vec{r}(t) = \langle \overset{x}{t^2}, \overset{y}{t^3} \rangle$

is not smooth at the point $t = 0$

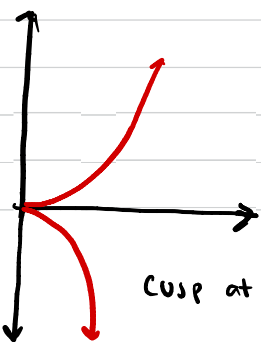
Since $\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$ and $\vec{r}\,'(0) = \langle 0, 0 \rangle = \vec{0}$

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}$$

$t = x^{1/2}$

$y = (x^{1/2})^3 = x^{3/2}$

$y = x^{3/2}$



Cusp at $t = 0$

if $\vec{r}(t)$ is the position vecter at time t $\vec{r}'(t) = v(t)$ (velocity)

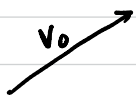
and $r''(t) = \vec{a}(t)$ (acceleration) and speed is $|v(t)|$

· length of cure $\vec{r}(t)$ when $t \in (t_1, t_2) = \displaystyle\int_{t_1}^{t_2} |\vec{v}(t)|\, dt$

Shows distance travelled from t_1 to t_2

· $$\int_{t_1}^{t_2} \sqrt{(x'(y))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

example: $\vec{F} = \vec{0}$ particle moving with an initial velocity $\vec{V}(0) = \vec{V}_0$



but zero acceleration $\vec{a} = \vec{0}$

$\vec{a} = 0$

$$V(t) = \int \vec{a}(t)\, dt = \int \vec{0}\, dt = \int \langle 0,0,0 \rangle\, dt$$

$$\int 0 + \int 0 + \int 0 = \langle c_1, c_2, c_3 \rangle$$

$\vec{v}(t) = \vec{c}$

at $t = 0$ $\vec{V}(0) = \vec{V}_0 = \vec{c}$

$r(t) = \int \vec{V_0}\, dt = \int \langle c_1, c_2, c_3 \rangle dt = \langle c_1 t + d_1, c_2 t + d_2, c_3 t + d_3 \rangle$

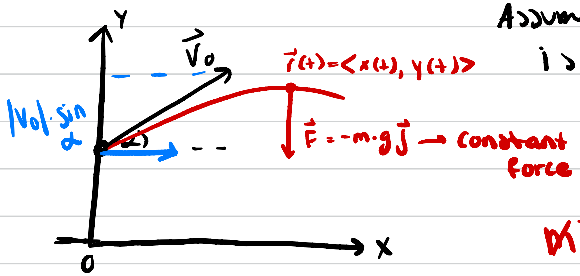
$= t\langle c_1, c_2, c_3 \rangle + \langle d_1, d_2, d_3 \rangle$

$\vec{r}(t) = t\vec{V_0} + \vec{d}$

at $t=0$ $\quad \vec{r}(0) = \vec{0} + \vec{d} = \vec{d} = \vec{r_0}$

$\vec{r}(t) = \vec{r_0} + t\vec{V_0}$ (equation of a line)

example 2: Projectile motion: Assume only gravity acts on the projectile



Assume at $t=0$, the initial velocity is $\vec{V_0}$

$m\vec{a} = -mg\vec{j} \Rightarrow \vec{a} = -g\vec{j} = \langle 0, -g \rangle$

$\vec{V}(t) = \int \vec{a}(t)\, dt = \int -g\vec{j}\, dt = -g(t)\vec{j} + \vec{C}$

$\vec{V}(0) = 0 \cdot \vec{j} + \vec{C} = \vec{C} = \vec{V}(0) = \vec{V_0}$

$\vec{V}(t) = -gt\vec{j} + \vec{V_0}$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int (-gt\vec{j} + \vec{v}_0)\,dt = \frac{-gt^2}{2}\vec{j} + \vec{v}_0 t + \vec{c}$$

$$\vec{r}(0) = \vec{r}_0 = \vec{c}$$

$$\vec{r}(t) = \frac{-gt^2}{2}\vec{j} + t\vec{v}_0 + \vec{r}_0$$

$$\vec{v}_0 = |\vec{v}_0|\cos\alpha\,\vec{i} + |\vec{v}_0|\sin\alpha\,\vec{j}$$

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

$$x(t)\vec{i} + y(t)\vec{j} = \frac{-gt^2}{2}\vec{j} + t\left(|v_0|\cos\alpha\,\vec{i} + |v_0|\sin\alpha\,\vec{j}\right) + \left(0\vec{i} + y_0\vec{j}\right)$$

$$\left.\begin{aligned} x(t) &= t|v_0|\cos\alpha \\ y(t) &= \frac{-gt^2}{2} + t|v_0|\sin\alpha + y_0 \end{aligned}\right\}$$

Parametric equation of the projectile.

↳ Graph will be a parabola.

14.2 more on computational rules for vector-valued functions.

- $\frac{d}{dt}\left(\vec{r}_1(t)+\vec{r}_2(t)\right)=\vec{r}_1\,'(t)+\vec{r}_2\,'(t)$

- $\frac{d}{dt}\left(f(t)\cdot\vec{r}(t)\right)=f'(t)\cdot\vec{r}(t)+f(t)\cdot\vec{r}\,'(t)$ (product rule)

Proof: $\frac{d}{dt}\left(f(t)\langle x(t), y(t), z(t)\rangle\right) =$

$\frac{d}{dt}\langle f(t)x(t), f(t)y(t), f(t)z(t)\rangle$

$\langle (f(t)x(t))' + (f(t)y(t))' + (f(t)z(t))'$

Do product rule

$= f'(t)\langle x(t), y(t), z(t)\rangle + f(t)\langle x'(t), y'(t), z'(t)\rangle$

- $\frac{d}{dt}\left(\vec{r}_1(t)\cdot\vec{r}_2(t)\right)=\vec{r}_1\,'(t)\cdot\vec{r}_2(t)+\vec{r}_1(t)\cdot\vec{r}_2\,'(t)$ (product rule)

- $\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right)=\vec{r}_1\,'(t)\times\vec{r}_2(t)+\vec{r}_1(t)\times\vec{r}_2\,'(t)$

(product rule)

- $\vec{r}(t)$ but $t = f(\alpha)$

  $\vec{r}(f(\alpha))$ * Geometrically its the same curve but with different parametrizations

- $\vec{r}(t)$ $t = f(\alpha)$ * Chain Rule

$$\frac{dr}{d\alpha} = \frac{d}{d\alpha}\left(\vec{r}(f(\alpha))\right) = \vec{r}\,'(f(\alpha)) \cdot (f'(\alpha))$$

ex: $\vec{r}(t) = \langle \cos t, \sin t \rangle \rightarrow$ circle

$$r'(t) = \langle -\sin t, \cos t \rangle$$

$$|r'(t)| = \sqrt{(\sin^2 t) + (\cos^2 t)} = \sqrt{1} = 1$$

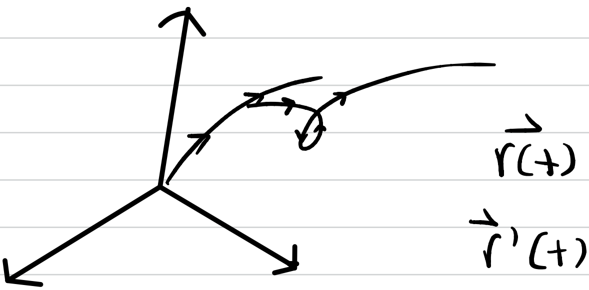
- $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $t = 5 \cdot \alpha$

$$\vec{r}(\alpha) = \langle \cos 5\alpha, \sin 5\alpha \rangle$$

$$\frac{d\vec{r}}{d\alpha} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{d\alpha} = \frac{d\vec{r}}{dt} \cdot 5 = 5\frac{d\vec{r}}{dt}$$

- $\frac{1}{f(t)} \cdot \vec{r}(t) = \left\langle \frac{x(t)}{f(t)}, \frac{y(t)}{f(t)}, \frac{z(t)}{f(t)} \right\rangle$

$$\frac{d}{dt}\left(\frac{1}{f(t)} \cdot r(t)\right) = \frac{d}{dt}\left(\frac{1}{f(t)}\right)\vec{r}(t) + \frac{1}{f(t)} \cdot \vec{r}\,'(t)$$

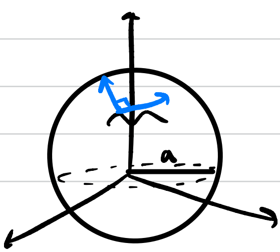
- 

$\vec{r}\,'(t)$ is a tangent vector to the curve.

$$\vec{T} = \frac{1}{|\vec{r}\,'(t)|} \cdot \vec{r}(t)$$

Unit tangent vector

ex: Suppose a particle is moving on a sphere centered at the origin with constant radius. Show that at every moment of time its velocity vector is perpendicular to its position vector.



$\vec{r}(t)$ is the position vector of the particle at time t.

Want to show that $\vec{r}\,'(t) \perp \vec{r}(t) \iff \vec{r}\,'(t) \cdot \vec{r}(t) = 0$

Know: $x^2+y^2+z^2=a^2$

$\downarrow$

equation of sphere with center $(0,0,0)$ and radius a.

$$(x(t))^2+(y(t))^2+(z(t))^2=a^2$$

$$|\vec{r}\,|^2=a^2$$. take $\frac{d}{dt}$

$$\frac{d}{dt}\left(|\vec{r}\,|^2\right)=\frac{d}{dt}\left(a^2\right) = 0$$

$$|\vec{r}(t)|^2=\vec{r}(t)\cdot\vec{r}(t)$$

$$\frac{d}{dt}\left(|\vec{r}(t)|^2\right)=\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right)$$

$$=\vec{r}\,'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}\,'(t)$$

$$2\left(\vec{r}\,'(t)\cdot\vec{r}(t)\right)$$

$$2\left(\vec{r}\,'(t)\cdot\vec{r}(t)\right)=0 \therefore$$

$$\vec{r}\,'(t)\perp r'(t)$$