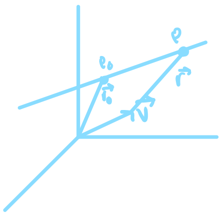
13.5 Lines and planes

1-25-22

A line is determined (in 3D or 2D space) by a point $P_0 \in L$ and a directional vector $\vec{v}$

$\vec{v} = \langle a, b, c \rangle$



$\vec{P_0P} = t \cdot \vec{v}$, t scalar

$\vec{r} - \vec{r_0} = t\vec{v}$

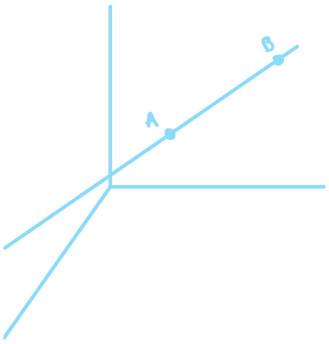
$\vec{r} = \vec{r_0} + t\vec{v}$ = vector equation of the line

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$x = x_0 + t_a$
$y = y_0 + t_b$
$z = z_0 + t_c$

Parametric equations of line L

Example: Find the parametric equation of a line that contains the points $A(0,7,1)$, $B(1,2,3)$



Take $\vec{v} = \vec{AB} = \langle 1, -5, 2 \rangle$

$x = 0 + t \cdot 1$
$y = 7 + t \cdot (-5)$
$z = 1 + t \cdot (2)$

$\Longleftrightarrow$

$x = t$
$y = 7 - 5t$
$z = 1 + 2t$

$\Downarrow$

$\vec{AD} = t \cdot \vec{AB}$

If $t \in [0,1]$, equation above describe all points on equation A, B

$t = 1/2$ get midpoint of AB

Position of lines and planes in 3D space

Exp 1: Determine if the given planes are parallel or intersecting. If they are intersecting determine the parametric equations of the intersecting line.

a) $\pi_1 = x + 2y + 3z = 4 \rightarrow \vec{n_1} = \langle 1, 2, 3 \rangle$

$\pi_1 \parallel \pi_2 \Leftrightarrow \vec{n_1} \parallel \vec{n_2} \Leftrightarrow \vec{n_2} = \overset{\text{scalar multiple}}{\hat{\alpha}} \cdot \vec{n_1}$

$\pi_2 = -3x - 6y - 9z = 13 \rightarrow \vec{n_2} = \langle -3, -6, -9 \rangle$

In our case $\vec{n_2} = -3\vec{n_1}$ therefore are parallel $\Rightarrow \pi_1 \parallel \pi_2$

or $\pi_1 = \pi_2$ ← in this case NO

$\Updownarrow$ divide by -3

$x + 2y + 3z = \frac{-13}{3}$

b) $x + 2y + 3z = 4$ $\quad \vec{n_1} = \langle 1, 2, 3 \rangle$
$2x + 4y - 3z = 0$ $\quad \vec{n_2} = \langle 2, 4, -3 \rangle$

not parallel to each other because not scalar multiples

So $\pi_1 \cap \pi_2 = L$ ← line in 3 space

$L: \begin{cases} x + 2y + 3z = 4 \\ 2x + 4y - 3z = 0 \end{cases}$ $\quad \begin{cases} x + 2y = 4 - 3z & (-2) \\ 2x + 4y = 3z \end{cases}$ $\Rightarrow 0 = -8 + 6z + 3z \quad z = 8/9$

$x + 2y = 4 - \frac{2 \cdot 8}{9 \, 3} = 4/3 \quad x = 4/3 - 2y \quad y = t$

$\qquad \qquad \qquad \qquad \qquad \qquad \parallel$

$\qquad \qquad \qquad \qquad \qquad 4/3 - 2t$

$L = \begin{cases} x = 4/3 - \cancel{2t} \\ y = 0 + \cancel{t} \\ z = 8/9 + \cancel{0t} \end{cases}$ Parametration of the line is not unique

$\searrow$

$\vec{v} = \cancel{\langle -2, 1, 0 \rangle}$

Example : Determine if the given line L intersect the given Plane π

$L = \begin{bmatrix} x = 1+2t \\ y = -3t \\ z = 2+t \end{bmatrix}$ $\pi = 3x + y - 2z = 9 \Rightarrow \vec{n}_1 = \langle 3, 1, -2 \rangle$ = normal vector to the plane

$\Downarrow$ $\vec{v}\,\langle 2, -3, 1 \rangle$ is directional vector for L

$L \parallel \pi \iff \vec{v} \parallel \pi \iff \vec{v} \perp \vec{n}_1 \iff \vec{v} \cdot \vec{n} = 0$

or $L \subset \pi$

dot Product $= 2\cdot 3 + (-3)\cdot 1 + 1\cdot(-2) = 1 \neq 0$

so $L \cap \pi = \{P\}$

this case the line intersects

To solve for the Point Plug in and solve system

$\begin{bmatrix} x = 1+2t \\ y = -3t \\ z = 2+t \\ 3x+y-2z=9 \end{bmatrix} \iff$ $3(1+2t) + (-3t) - 2(2+t) = 9$

$t = 9+1 = t = 10$

plug in to equation

$P(21, -30, 12)$

if $\vec{v}$ is a Scalar multiple of $\vec{n}$ then they are Perpendicular intersection

c) What angle does the line intersect the plane?

Let y be the angle between $\vec{n}$ and $\vec{v}$

$$\boxed{\cos y = \frac{\vec{v} \cdot \vec{n}}{|\vec{v}|\,|\vec{n}|}}$$

important

$\cos y = \frac{1}{\sqrt{2^2+3^2+1^2}} = \cos y = \frac{1}{\sqrt{14}\cdot\sqrt{14}}$

$\Downarrow$

$90^\circ - 85.9^\circ \Leftarrow \cos^{-1}(1/14) = 85.9 \Leftarrow \frac{1}{14} = y$

$\Downarrow$

$\theta = 4.1^\circ$

Example : Given the line L_1 and L_2 decide if they are parallel, intersecting or screwed, or the same

L_1: $x = 1+2t$
$y = -3t$
$z = 2+t$

$\Downarrow$

$\vec{v}_1 = \langle 2, -3, 1 \rangle$

L_2: $x = 2-t$
$y = 1+3t$
$z = 0$

$\Downarrow$

$\vec{v}_2 = \langle -1, 3, 0 \rangle \Rightarrow$ not Parallel because they are not scalar multiples

To check if they intersect **important to change the name of the parameter for one of them** : $L_1 \begin{bmatrix} x = 1+2t \\ y = -3t \\ z = 2+t \end{bmatrix}$ $L_2 \begin{bmatrix} x = 2-s \\ y = 1+3s \\ z = 0 \end{bmatrix} \Rightarrow$

$1+2t = 2-s \Rightarrow 2-s = -3$ $\cancel{s=5}$

$-3t = 1+3s$ $-3(-2) = 1+3\cdot 5 = 16 \leftarrow$ not satisfied ~~therefore L_1 and L_2 are screwed~~

$2+t = 0 \Rightarrow \cancel{t=-2}$

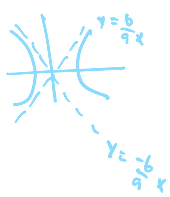
13.6 Quadric Surface in 3 space

General equation: $ax^2 + by^2 + cz^2 + \underbrace{dxy + exz + fyz + gx + hy + kz}_{\text{Standard form removes these}} + l = 0$

Conic-Curves in the 2 plane

1 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ = ellipse

2 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ hyberbola



3. $y = 2kx^2 + c$
or
$x = 2ky^2 + c$

4. degenerate Conic :

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$] Pair of two lines

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ a dot $(0,0)$

Types of Quadric surfaces

$\frac{x^2}{y^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoid $\rightarrow$

$F \cap \{z=0\} \rightarrow$ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$F \cap \{z=k\}$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$