Local Linear approximation (or lineritation) of a function $f(x,y)$ at point $(x_0,y_0)$

$$f(x,y) \approx \underbrace{f(x_0,y_0) + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y}_{L(x,y)}$$

$L(x,y)$ ← the lineritation of $f(x,y)$ at $(x_0,y_0)$

approximation is good when $(x,y)$ is close to $(x_0,y_0)$

$$z = z_0 + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$$

Equation of tangent Plane to the graph of $z=f(x,y)$ at $(x_0,y_0)$

$z_0 = f(x_0,y_0)$

Ex: $f(x,y) = xe^{x^2-y}$

$f(1,1) = 1\cdot e^0 = 1$

a) Find the local linear approximation for $f(x,y)$ near $(1,1)$

b) Use the approximation found to estimate without calculator $f(1.02, 0.99)$

c) Find the lineritation and the equation of the tangent plane to the graph of $z=f(x,y)$ at $(1,1)$

$$f_x = \frac{\partial}{\partial x}\left(x\cdot e^{x^2-y}\right) = 1\cdot e^{x^2-y} + xe^{x^2-y}\cdot 2x = e^{x^2-y}\left(1+2x^2\right)$$

$$f_x(1,1) = e^0\left(1+2\cdot 1^2\right) = 3$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x\cdot e^{x^2-y}\right) = x\cdot e^{x^2-y}(-1) = -xe^{x^2-y}$$

$$f_y(1,1) = -1$$

Local linear approximation $= x\cdot e^{x^2-y} \approx 1 + 3(x-1) - 1(y-1)$

b) $f(1.02, 0.99) \approx 1 + 3(1.02-1) - 1(0.99-1) = 1 + 0.06 + 0.01 \approx 1.07$

c) $L(x,y) = 1 + 3(x-1) - 1(y-1)$ ; Tangent plane $z = 1 + 3(x-1) - 1(y-1)$

(Total) differential of a funtion $z=f(x,y)$ is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$$f(x,y) - f(x_0,y_0) = \Delta z \approx dz$$

$\Delta z$: actual change in $z$

$dz$: change in $z$ given by the differential

Ex: Volume of a cone $V(r,h) = \frac{1}{3}\pi r^2\cdot h$

a) Use differential to estimate the change in volume of a cone when the radius is increase from $r=10$ to $r=10.2$ and height is decreased from $h=10$ to $h=9.7$

b) Use differentials to estimate the percentage change in volume if radius increases by 2% and height decreases by 5%.

a) $dv = \frac{dv}{dr}dr + \frac{dv}{dh}dh$ $\Rightarrow$ $dv = \frac{2}{3}\pi r h\, dr + \frac{1}{3}\pi r^2\, dh$

$\frac{dv}{dr} = \frac{2}{3}\pi r h$ $\quad \frac{dv}{dh} = \frac{1}{3}\pi r^2$ $\quad r=10\ h=10\quad \Delta r = dr = 0.2\quad \Delta h = dh = -0.3$

$dv(10,10) = \frac{2}{3}\pi\, 10\cdot 10\,(0.2) + \frac{1}{3}\pi\cdot 10^2(-0.3) = \frac{40\pi}{3} - \frac{30\pi}{3} = \frac{10\pi}{3}$

$\Delta V \approx dv = \frac{10\pi}{3}$

b) Percentage change of a quantity $Q = \frac{\Delta Q}{Q}$

Percentage change of volume $V = \frac{\Delta V}{V} = -0.01$

$\frac{dr}{r} = \frac{\Delta r}{r} = 0.02 = 2\%$

$\frac{dh}{h} = \frac{\Delta h}{h} = -0.05 = -5\%$ $\qquad \frac{2dr}{r} + \frac{dh}{h} \Rightarrow 2\cdot 0.02 - 0.05 = -0.01$

$\frac{\Delta V}{V} \approx \frac{dv}{V} = \frac{\frac{2}{3}\pi r h\cdot dr + \frac{1}{3}\pi r^2 dh}{\frac{1}{3}\pi r^2 h} = \frac{\frac{2}{\cancel{3}}\cancel{\pi r h}\, dr}{\cancel{\frac{1}{3}\pi r^2 h}} + \frac{\cancel{\frac{1}{3}\pi r^2}\, dh}{\cancel{\frac{1}{3}\pi r^2}\, h}$

## Chain Rule 15.4

Most basic scenario for $f(x,y,z)$

$z = f(x,y) \longrightarrow \Delta z \approx dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$

but $x = x(t)$ and $y = y(t)$

$z(t) = f(x(t), y(t)) \Rightarrow \frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$

$\lim_{\Delta t\to 0} \frac{\Delta z}{\Delta t} \Rightarrow \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$

More general cases of Chain Rule

$w = f(x,y,z)$

but $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

$w(u,v) = f(x(u,v), y(u,v), z(u,v))$

$\frac{dw}{du} = \frac{dw}{dx}\cdot\frac{dx}{du} + \frac{dw}{dy}\cdot\frac{dy}{du} + \frac{dw}{dz}\cdot\frac{dz}{du}$

$\frac{dw}{dv} = \frac{dw}{dx}\cdot\frac{dx}{dv} + \frac{dw}{dy}\cdot\frac{dy}{dv} + \frac{dw}{dz}\cdot\frac{dz}{dv}$

$\left.\frac{dz}{dt}\right|_{t=0}$ $\quad z = x^2 y$ and $x = e^t$ $\quad y = \cos t$

$\Downarrow$

$\frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}$

$\Downarrow$

$2xy\,(e^t) + x^2 - \sin(t)$

$\frac{dz}{dt} = 2xye^{t} - x^2(\sin(t)) = 2e^{t}\cos(t)e^{t} - e^{2t}\sin(t)$

$= 2e^0\cos(0) - e^0\sin(0) = 2$

Ex: Suppose $z$ is implicitely defined as a function of $(x,y)$ by the Relation $F(x,y,z) = C$

Find a formula for $\frac{dz}{dx}$ and $\frac{dz}{dy}$ ?

b) Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ if $z$ is implicitely given by $x^2z - x^2y + \sin(z) = 10$

be done with implicit differentation

Take $\frac{d}{dx}$ of both sides of $f(t)$

$$\frac{d}{dx}\left(x^2z - x^2y + \sin(z)\right) = \frac{d}{dx}(10)$$

$$2xz + x^2\frac{dz}{dx} - 2xy + \cos(z)\frac{dz}{dx} = 0$$

$$\frac{dz}{dx}\left(x^2 + \cos(z)\right) = 2xy - 2xz$$

$$\Downarrow$$

$$\frac{dz}{dx} = \frac{2xy - 2xz}{x^2 + \cos(z)}$$

Maybe Proof on Test

A) $F(x,y,z) = C \rightarrow$ gives implicitely $z = z(x,y)$

$F(x,y,z(x,y)) = C \quad (*)$

Take $\frac{d}{dx}$ of both sides of $(*)$

$$= \frac{d}{dx}\left(F(x,y,z(x,y))\right) = 0$$

$$\frac{dF}{dx}\cdot\underbrace{\frac{dx}{dx}}_{1} + \cancel{\frac{dF}{dy}\cdot\underbrace{\frac{d(y)}{dx}}_{0}} + \frac{dF}{dz}\cdot\frac{d(z)}{dx} = 0 \Rightarrow \frac{dF}{dx} + \frac{dF}{dz}\cdot\frac{dz}{dx} = 0 \Rightarrow -\frac{\frac{dF}{dx}}{\frac{dF}{dz}} = \frac{dz}{dx}$$

$F(x,y,z) = x^2z - x^2y + \sin(z)$

$\frac{dF}{dx} = 2xz - 2xy + 0$ apply

$\frac{dF}{dz} = x^2 + \cos(z)$ $= \frac{dz}{dx} = -\frac{2xz - 2xy}{x^2 + \cos(z)} = \frac{-2xz + 2xy}{x^2 + \cos(z)}$