directional derivative and gradient

$f(x,y)$ "mabla" (Gradient of f)

$(\nabla f)(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\hat{i} + \frac{\partial f}{\partial y}(x_0,y_0)\hat{j} =$
$\langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle$

$(\nabla f)(x_0,y_0,z_0) = \langle f_x(x_0,y_0,z_0) + f_y(x_0,y_0,z_0) + f_z(x_0,y_0,z_0) \rangle$

DIRECTIONAL DERIVATIVE

$f(x,y)$ suppose $\vec{u}$ is a unit vector. $\vec{u} = \langle u_1, u_2 \rangle$

$$\begin{cases} x = x_0 + su_1 \\ y = y_0 + su_2 \end{cases}$$

$f(x_0+su_1, y_0+su_2)$

Definition

$(D_{\vec{u}}f)(x_0,y_0) \underline{\text{def}} \frac{d}{ds}(f(x_0+su_1, y_0+su_2)\Big|_{s=0} =$ *

Directional derivative of in the direction $\vec{u}$ at (x_0,y_0)

chain rule

* $\frac{\partial f}{\partial x}(x_0,y_0) \cdot \frac{d(x_0+su_1)}{ds} + \frac{\partial f}{\partial y}(x_0,y_0) \cdot \frac{d(x_0+su_2)}{ds}$

$= \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2 = \nabla f \cdot \vec{u}$

Directional derivative of f at (x_0, y_0) in direction $\vec{u}$

$(D_{\vec{u}} f)(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

- Let $T(x, y) = 30 + x^2 - 5y^2 + 2x$ be the temperature on a plate in °C at the point (x, y) (where x, y are measured in cm)

Suppose an ant is at a point $(2, 1)$

(a) Temperature at the point?

$T(2,1) = 30 + 2^2 - 5 \cdot 1^2 + 2 \cdot 2 = 33°C$

(b) Suppose the ant decides to move in the direction of vector $\vec{v} = \hat{i} + \hat{j}$ -> normalize $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$

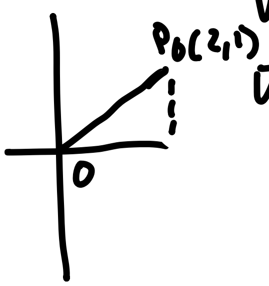
What rate of change of temperature does it experience?

$(D_{\vec{u}} T)(2,1) = (\nabla T)(2,1) \cdot \vec{u} = \langle 6, -10 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle =$

$\frac{1}{\sqrt{2}} \langle 1, 1 \rangle = \frac{1}{\sqrt{2}}(-4)$ -> dot product between $\langle 6, -10 \rangle \cdot \langle 1, 1 \rangle$

$D_{\vec{u}} T(2,1) = -\frac{4}{\sqrt{2}}$ °C / cm

(c) If the ant decides to go from (2,1) directly towards the origin. What is the rate of change of temperature that the ant experiences as it starts this trip?



$$\vec{v} = \overrightarrow{P_0 O} = \langle -2, -1 \rangle$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{5}}\langle -2, -1 \rangle$$

$$D_{\vec{u}}T = (\nabla T)(2,1) \cdot \vec{u} = \langle 6, -10 \rangle \cdot \frac{1}{\sqrt{5}} \langle -2, 1 \rangle$$

$$= \frac{1}{\sqrt{5}}(-12 + 10) = \frac{2}{\sqrt{5}}$$

(d) In which direction should the ant go from (2,1) to experience the biggest increase in temperature?

Important: (theorem 1)

$\nabla f(x_0, y_0)$ gives the direction in which the function increases most rapidly

$$\vec{u} = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$ $\leftarrow$ direction of the most rapid increase.

$\vec{u} = -\left(\frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}\right) \leftarrow$ direction of the most rapid decrease

back to the question:

- $\vec{u}$ should have the same direction as $\nabla T(2,1)$

$$\vec{u} = \frac{\nabla T(2,1)}{|\nabla T(2,1)|} = \frac{\langle 6, -10 \rangle}{\sqrt{(6)^2 + (-10)^2}} = \frac{1}{\sqrt{36}} \langle 6, -10 \rangle$$

What is the rate of change of temperature in this direction?

$|\nabla T(2,1)| = \sqrt{136}$ °C/cm

e) Suppose the ant is a heat seeking ant.
Can you find its trajectory?

$\vec{r}(t) = (x(t), y(t))$

$\uparrow$ position vector of the ant at time t.

Heat-seeking ant

$\vec{r}'(t) = k(t) \cdot \nabla T(x(t), y(t)) \quad k > 0$

$\langle x'(t), y'(t) \rangle = k \langle 2x(t) + 2, -10y(t) \rangle$

$$\begin{cases} x'(t) = k(t)(2x(t)+2) \\ y'(t) = k(t)(-10y(t)) \end{cases}$$

Take the ratio of these to get rid of k

$$\frac{y'(t)}{x'(t)} = \frac{-10y}{2x+2} \qquad \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\boxed{\frac{dy}{dx} = \frac{-10y}{2x+2}}$$

$$\frac{dy}{-10y} = \frac{dx}{2x+2} \rightarrow \text{factor this}$$

$$-\frac{1}{10}\int \frac{dy}{y} = \frac{1}{2}\int \frac{dx}{x+1}$$

$$-\frac{1}{10}\ln y = \frac{1}{2}\ln|x+1| + c$$

Initial point $(2,1)$ $x=2, y=1$

$$-\frac{1}{10}\ln 1 = \frac{1}{2}\ln 3 + c$$

$$c = \frac{1}{2}\ln 3$$

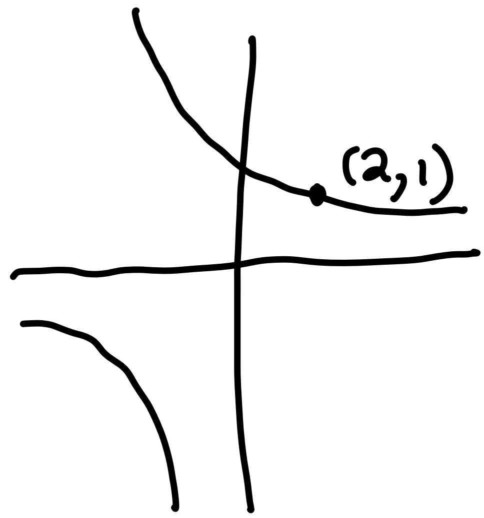
$$-\frac{1}{10}\ln y = -\frac{1}{2}(x+1) - \frac{1}{2}\ln(3) \;\Big|\; (-10)$$

$$\ln y = -\frac{10}{2}\left(\ln(x+1) - \ln(3)\right)$$

$$\ln y = \ln\left(\left(\frac{x+1}{3}\right)^5\right)$$

$$y = \left(\frac{x+1}{3}\right)^{-5}$$

$$y = \frac{3^5}{(x+1)^5}$$



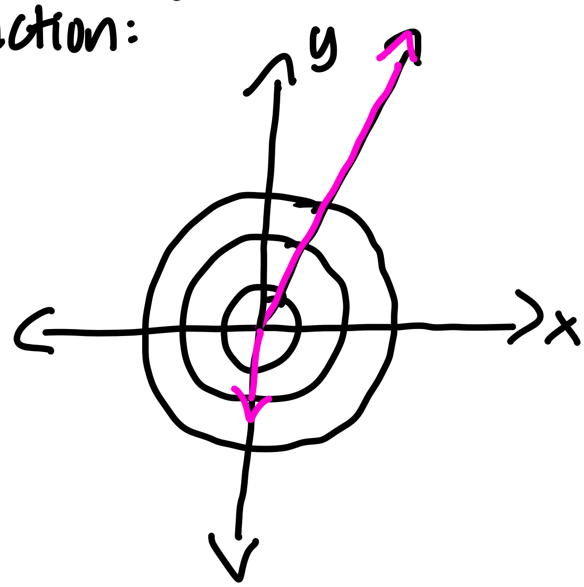
Theorem #2:

The gradient of $\nabla f(x,y)$ is always normal to the level curves of the function:

$z = f(x,y) = x^2+y^2$

Level curves $k = x^2+y^2$

$\nabla f = \langle 2x, 2y\rangle = 2\langle x, y\rangle$

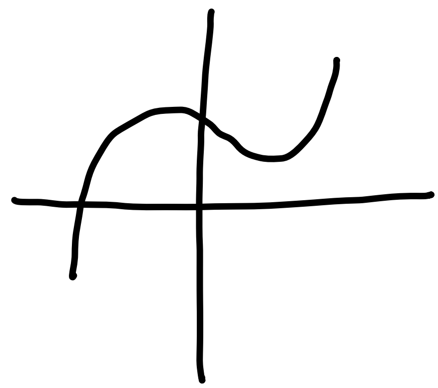


Proof of theorem 2:

$f(x,y) = k \leftarrow k =$ level curve

↳ represent this curve in a parametric way as $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\frac{d}{dt}\left(f(x(t), y(t))\right) = 0$$



↳ chain rule

$$\frac{\partial f}{\partial t} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle = 0$$

$$\nabla f \cdot \vec{r}\,'(t) = 0 \Longleftrightarrow \nabla f \perp \vec{r}\,'(t)$$

∇f is normal to the level curve

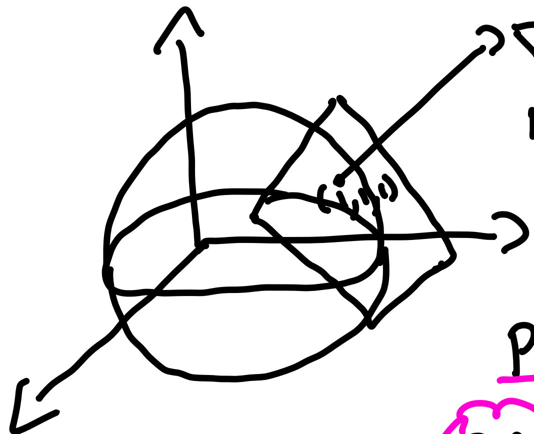
Theorem 2: a) The gradient $\nabla f(x,y)$ is always normal to the level curves of the function.

b) $f(x,y,z)$ The gradient $\nabla f(x,y,z)$ is always normal to the level surfaces of the function $f(x,y,z)$

Example: Find the tangent plane to the ellipsoid $x^2+2y^2+3z^2=6$ at the point $(1,1,1)$

Let $w=F(x,y,z)=x^2+2y^2+3z^2$

The given ellipsoid is a level surface for $F(x,y,z)$



$\nabla F(x,y,z)$

$\vec{n}=\nabla F(1,1,1)$

$\nabla F(x,y,z)=\langle 2x,4y,6z\rangle$

$\nabla F(1,1,1)=\langle 2,4,6\rangle=\vec{n}$

Point normal

$2(x-1)+4(y-1)+6(z-1)=0$

equation of the tangent plane of the ellipsoid.