Quiz: Thursday from section 15.5 directional derivatives

Exam 2: Chapter 15

15.5 Directional derivatives and gradient

$f(x,y)$

vector

$$\nabla f\,(x_0,y_0) = \frac{df}{dx}(x_0,y_0)\,\vec{i} + \frac{df}{dy}(x_0,y_0)\,\vec{j} = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle$$

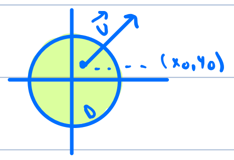
"nabla"

partial derivative

$$f(x,y,z) \Rightarrow (\nabla f)(x_0,y_0,z_0) = \langle f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0)\rangle$$

Directional derivative

$f(x,y)$



Suppose $\vec{v}$ is a unit vector

$\vec{v} = \langle v_1, v_2\rangle$

$x = x_0 + s v_1 \Rightarrow f(x_0 + s v_1,\ y_0 + s v_2)$

$y = y_0 + s v_2$

chain rule

Definition: $D_{\vec{v}} f(x_0,y_0) = \frac{d}{ds}\left(f(x_0+sv_1, y_0+sv_2)\right)\Big|_{s=0} = \frac{df}{dx}(x_0,y_0)\cdot\frac{d(x_0+sv_1)}{ds} + \frac{df}{dy}(x_0,y_0)\cdot\frac{d(y_0+sv_2)}{ds}$

Directional derivative of in the direction $\vec{v}$ at (x_0,y_0)

$$\Downarrow$$

$$\frac{df}{dx}\cdot v_1 + \frac{df}{dy}\cdot v_2 = \nabla f\cdot\vec{v}$$

dot Product

Directional derivative of f at (x_0,y_0) in direction $\vec{v}$

$$(D_{\vec{v}} f)(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{v}$$

Ex: Let $T(x,y) = 30 + x^2 - 5y^2 + 2x$

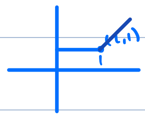
Let the temperature on a plate in C at the point (x,y) (where x,y are measured in cm)

Suppose an ant is at the point $(2,1)$

A) What is the temperature at that point

$T(2,1) = 30 + (2)^2 - 5(1)^2 + 2(2) = 33^\circ C$

B) Suppose the ant decides to move in the direction of vector $\vec{v} = \hat{i} + \hat{j}$ what rate of change of temperature does it experience



Normalize $\vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}\langle 1,1\rangle = \vec{u}$

b) $D_{\vec{u}}T(2,1) = ?$

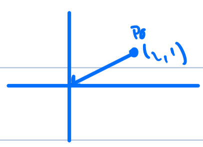
$\nabla T = \langle \frac{dT}{dx}, \frac{dT}{dy} \rangle \Rightarrow \langle 2x+2, -10y \rangle$

$\nabla T(2,1) = \langle 6, -10 \rangle$

$D_{\vec{u}}T(2,1) = \langle 6, -10\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1\rangle = \frac{1}{\sqrt{2}} - 4 = \frac{-4}{\sqrt{2}}$ celcius/cm

Experencing decrease in Temp

C) Suppose ant goes straight to the origin what is the rate of change of temperature that the ant experiences as it starts this journey



$\vec{v} = \vec{P_0 O} = \langle -2, -1\rangle$

normalize $\vec{u} = \frac{\vec{v}}{|\vec{v}|} \Rightarrow \frac{1}{\sqrt{5}}\langle -2,-1\rangle \Rightarrow$

$|\vec{v}| = \sqrt{2^2 + 1^2}$

$D_{\vec{u}}T = (\nabla T)_{(2,1)} \cdot \vec{u} = \langle 6,-10\rangle \cdot \frac{1}{\sqrt{5}}\langle -2,-1\rangle = \frac{1}{\sqrt{5}}(-12+10) \Rightarrow -\frac{2}{\sqrt{5}}$

D) In which direction should the ant go from (1,1) to experience the biggest increase in temperature

$|\nabla f(x_0,y_0)| - |\vec{u}| \cdot \cos\theta$] max when $\theta = 0$

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Theorem 1:

Important Conclusion: $\nabla f(x_0,y_0)$ gives the direction in which the function increases most rapidly

$\Downarrow$

$\vec{u} = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$ direction of most rapid increase

Theorem 2: The gradient $\nabla f(x,y)$ is always normal to the level curves of the function

$\vec{u} = -\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$ direction of most rapid decrease

Rate of change: $|\nabla f(x_0, y_0)|$

D): $\vec{v}$ should have same direction as $\nabla T(2,1)$

$\vec{v} = \frac{\nabla T}{|\nabla T|} \Rightarrow \frac{\langle 6,-10\rangle}{\sqrt{6^2+10^2}} \Rightarrow \frac{1}{\sqrt{136}}\langle 6,-10\rangle$

rate of change: $|\nabla T(2,1)| = \sqrt{136}$ c/cm

$z = f(x,y) = x^2+y^2 \Rightarrow$

Level curves: $k = x^2+y^2$

$\Downarrow$

$\nabla f = \langle 2x, 2y\rangle$

$= 2\langle x, y\rangle$

E) Suppose the ant is a heat seeking ant can you find its trajectory

$\vec{r}(t) = \langle x(t), y(t)\rangle$ $\qquad \vec{r}(0) = \langle 2, 1\rangle$

$\Downarrow$

Position vector of ant at time t

Heat seeking ant

$$\vec{r}\,'(t) = k \cdot \nabla T(x(t), y(t))$$

$$\downarrow$$

$$\langle x'(t), y'(t)\rangle = k \cdot \langle 2x(t)+2, -10y(t)\rangle$$

$\nabla T(x,y) = \langle 2x+2, -10y\rangle$

$$\left\{\begin{array}{l} x'(t) = k(2x(t)+2) \\ y'(t) = k(-10y(t)) \end{array}\right.$$ take ratio get rid of k

$$\Downarrow$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} = \frac{-10y}{2x+2} \Rightarrow \frac{dy}{dx} = -\frac{10y}{2x+2}$$

$$\Downarrow$$

$$\frac{dy}{-10y} = \frac{dx}{2x+2}$$

$$\Downarrow$$

$$\frac{-1}{10}\int \frac{dy}{y} = \frac{1}{2}\int \frac{dx}{x+1}$$

initial point is $(2,1)$ $\quad x=2, y=1$

Plug in

$$-\frac{1}{10}\ln y = \frac{1}{2}\ln(x+1) + C$$

$$\frac{-1}{10}\ln 1 = \frac{1}{2}\ln 3 + C \Rightarrow C = \frac{-1}{2}\ln 3$$

$$\cancel{\frac{-1}{10}}\ln y = \left(\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln 3\right)\cdot 10$$

$$\Downarrow$$

$\ln y = -\frac{10}{2}(\ln(x+1) - \ln 3) \Rightarrow \ln y = (\ln(\frac{x+1}{3})^{5}) = y = (\frac{x+1}{3})^{-5}$

Proof of Theorem 2

a) $f(x,y) = k \leftarrow$ k level curve

$\downarrow$

represent this curve in a parametric way as $\vec{r}(t) = \langle x(t), y(t) \rangle$

$f(x(t), y(t)) = k$] take $\frac{d}{dt}$ both sides

$\frac{d}{dt}(f(x(t), y(t))) = 0$

$\downarrow$ Chain rule

$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0 \Leftrightarrow \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \langle x'(t), y'(t) \rangle = 0$

$\nabla f \cdot \vec{r}'(t) = 0 \Leftrightarrow \nabla f \perp \vec{r}'(t)$

$\Updownarrow$

$\uparrow$ Should know proof

∇f is normal to the level curve

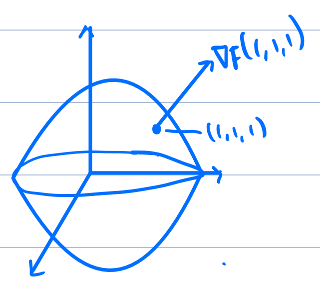
b) $f(x,y,z) = k \leftarrow$ level surfaces

The gradient $\nabla f(x,y,z)$ is always normal to the level surfaces of the function $f(x,y,z)$.

Ex: Find the Tangent plane to the ellipsoid $\overbrace{x^2 + 2y^2 + 3z^2}^{F(x,y,z)} = 6$ at the point $(1,1,1)$

Let $w = F(x,y,z) = x^2 + 2y^2 + 3z^2$

The ellipsoid is a level surface for $F(x,y,z)$



$\vec{n} = \nabla F(1,1,1)$

$\nabla F(x,y,z) = \langle 2x, 4y, 6z \rangle$

$\nabla F(1,1,1) = \langle 2, 4, 6 \rangle = \vec{n}$

Point normal: $2(x-1) + 4(y-1) + 6(z-1) = 0$

$\uparrow$ equation of tang plane