NAME: Solution Key Panther ID: ____________

Exam 2 - MAC 2313 Spring 2022

**Important Rules:**

**A. Any electronic device (cell phone, calculator of any kind, smart-watch, etc.) should be turned off at the beginning of the exam and placed in your bag, NOT in your pocket. Electronic items, notes, texts, or formula sheets should NOT be used at any time during the examination. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.**

**Cheating attempts will lead to a score of zero on this exam, and possibly a report for academic misconduct.**

**B. Unless otherwise mentioned, to receive full credit you must show your work. Answers which are not supported by work might receive no credit. Solutions should be concise and clearly written. Incomprehensible work might not be considered.**

**1.** (14 pts) Consider the function $f(x, y) = x^3 + xe^{-3y}$.

(a) (6 pts) Find the partial derivatives $f_x$ and $f_y$.

$f_x = 3x^2 + e^{-3y}$ $\quad$ $f_y = -3xe^{-3y}$

(b) (4 pts) Find the linearization (or local linear approximation) of the function $f(x, y)$ near the point $P_0 = (1, 0)$.

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f(1,0) = 1 + 1\cdot e^0 = 2$ $\quad$ $f_x(1,0) = 3\cdot 1^2 + e^0 = 4$ $\quad$ $f_y(1,0) = -3$

Thus $L(x,y) = 2 + 4(x-1) - 3(y-0)$ ← the linearization of $f(x,y)$ near (1,0

(c) (4 pts) Use your answer in part (b) to estimate the value of $f(0.99, 0.02)$.

$f(x,y) \approx 2 + 4(x-1) - 3y$ ← local lin. approximation near $(1,0)$.

$f(0.99, 0.02) \approx 2 + 4(0.99-1) - 3\cdot(0.02) = 2 - 0.04 - 0.06 = 1.9$

Thus, $f(0.99, 0.02) \approx 1.9$

**2.** (12 pts) Circle if each of the following statements is true or false. No justification needed.

(a) The domain of the function $f(x,y) = \sqrt{1-x^2-y^2}$ is the whole plane $\mathbf{R}^2$. **True** **False**

(b) If $f_x$ and $f_y$ exist at $P$, then $f(x,y)$ is differentiable at $P$. **True** **False**

(c) If $f_{xy}$ and $f_{yx}$ exist and are continuous, then $f_{xy} = f_{yx}$. **True** **False**

(d) The set $R = \{(x,y) \mid -2 \le y \le 2\}$ is bounded in $\mathbf{R}^2$. **True** **False**

(e) The set $R = \{(x,y) \mid -2 \le y \le 2\}$ is closed in $\mathbf{R}^2$. **True** **False**

(f) Any continuous function defined on the set $R = \{(x,y) \mid -2 \le y \le 2\}$ has an absolute maximum and an absolute minimum on $R$. **True** **False**

**3.** (10 pts) Find an equation of the tangent plane to the surface $x^2 + y^3 + z^4 = 2$ at the point $(-1, 0, 1)$.

If $F(x,y,z) = x^2 + y^3 + z^4$, the surface $x^2+y^3+z^4=2$ is a level surface for $F(x,y,z)$.

Thus, a normal vector to the surface at $(-1,0,1)$ is $(\nabla F)(-1,0,1)$

$\nabla F = \langle 2x, 3y^2, 4z^3 \rangle$ so

$\vec{n} = (\nabla F)(-1,0,1) = \langle -2, 0, 4 \rangle$

Tangent plane has equation:

$-2(x-(-1)) + 4(z-1) = 0$ or $\boxed{-2(x+1) + 4(z-1) = 0}$

or $\underline{-x + 2z = 3}$

**4.** (14 pts) Consider the function $f(x, y) = \ln(1 + 3x + 4y)$.

(a) (6 pts) Find the gradient $\nabla f$ at $(0, 0)$.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{3}{1+3x+4y}, \frac{4}{1+3x+4y} \right\rangle$$

$$(\nabla f)(0,0) = \langle 3, 4 \rangle$$

(b) (4 pts) Find the directional derivative of $f$ at the point $(0, 0)$ in the direction of the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ (note that $\mathbf{a}$ is not a unit vector).

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{5}}(\vec{i} - 2\vec{j})$$

$$(D_{\vec{u}} f)(0,0) = (\nabla f)(0,0) \cdot \vec{u} = \langle 3,4 \rangle \cdot \frac{1}{\sqrt{5}} \langle 1,-2 \rangle = \frac{1}{\sqrt{5}}(3-8) = -\frac{5}{\sqrt{5}} = -\sqrt{5}$$

(c) (4 pts) Find a unit vector in the direction in which $f$ decreases most rapidly at $(0, 0)$.

The direction of most rapid decrease is given by $-(\nabla f)(0,0)$

Thus $$\vec{u} = -\frac{(\nabla f)(0,0)}{|(\nabla f)(0,0)|} = -\frac{1}{\sqrt{3^2+4^2}} \langle 3,4 \rangle = \boxed{-\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}$$

**5.** (12 pts) Show that the function $u(x,t) = e^{-a^2 t}\Big(B\cos(ax) + C\sin(ax)\Big)$ satisfies the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

for any constants $a$, $B$, $C$.

$$\frac{\partial u}{\partial t} = -a^2 \cdot e^{-a^2 t}\left(B\cos(ax) + C\sin(ax)\right)$$

$$\frac{\partial u}{\partial x} = e^{-a^2 t}\left(-Ba\sin(ax) + C\cdot a\cos(ax)\right)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-a^2 t}\left(-Ba^2\cos(ax) - Ca^2\sin(ax)\right) = -a^2 e^{-a^2 t}\left(B\cos(ax) + C\sin(ax)\right)$$

Thus, indeed $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

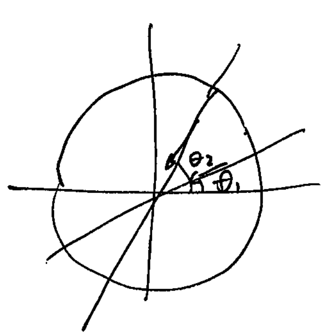
**6.** (10 pts) If the following limit exists, compute it. If the limit does not exist, justify why it doesn't.

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$$

Using polar coordinates (you could also investigate the limit across paths $y = mx$)

We have

$$\frac{3xy}{x^2+y^2} = \frac{3r\cos\theta \cdot r\sin\theta}{r^2} = 3\sin\theta\cos\theta$$




As approaching $(0,0)$ on different angles $\theta$ would procude a different outcome value, the given limit does not exist.

**7.** (14 pts) Find all critical points of the function $f(x,y) = x^4 + y^2 - 4xy$. Determine the type of each critical point (local maximum, local minimum, or saddle point).

$$\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 2y - 4x = 0 \end{cases} \iff \begin{cases} y = x^3 \\ y = 2x \end{cases}$$

So $x^3 = 2x$ or $x(x^2-2) = 0 \Rightarrow x = 0$, or $x = \pm\sqrt{2}$

We have three critical points

$P_1(0,0)$, $P_2(\sqrt{2}, 2\sqrt{2})$, $P_3(-\sqrt{2}, -2\sqrt{2})$

For the type of the critical points, we compute the Hessian

$$(\text{Hess} f)(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & 4 \\ 4 & 2 \end{pmatrix}$$

$$(\text{Hess} f)(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix} \qquad \Delta = \det(\text{Hess} f(0,0)) = -16 < 0$$

Thus $P_1(0,0)$ is a saddle point

$$(\text{Hess} f)(\sqrt{2}, 2\sqrt{2}) = \begin{pmatrix} 24 & 4 \\ 4 & 2 \end{pmatrix} \qquad \Delta = 32 > 0$$

$f_{xx} > 0$

Thus $P_2(\sqrt{2}, 2\sqrt{2})$ is a local minimum.

$$(\text{Hess} f)(-\sqrt{2}, -2\sqrt{2}) = \begin{pmatrix} 24 & 4 \\ 4 & 2 \end{pmatrix} \qquad \Delta = 32 > 0$$

$f_{xx} = 24 > 0$

Thus $P_3(-\sqrt{2}, -2\sqrt{2})$ is also a local min.

**8.** (12 pts) Find the extreme values of $f(x,y) = xy^2$, subject to the constraint $\underbrace{x^2+y^2}_{g(x,y)} = 1$. Lagrange multipliers method is recommended, but other correct solutions will receive full credit.

Final answer: Extreme values of $f$ are $f_{min} = -\frac{2}{3\cdot\sqrt{3}}$ $f_{max} = \frac{2}{3\sqrt{3}}$

$$\nabla f = \lambda \nabla g$$

$$\langle y^2, 2xy\rangle = \lambda\langle 2x, 2y\rangle$$
$$x^2+y^2=1$$

$$\Leftrightarrow \begin{cases} y^2 = 2\lambda x \\ \cancel{2}xy = \cancel{2}\lambda y \\ x^2+y^2=1 \end{cases}$$

$xy = \lambda y \to xy - \lambda y = 0 \Rightarrow y(x-\lambda) = 0$ — Case 1 $y=0$; Case 2 $x=\lambda$.

Case 1: $y=0 \to$ plug this in the $3^{rd}$ equation to get $x^2=1$

so we get the potential extremum points $(-1,0)$, $(1,0)$

(from $1^{st}$ equation it follows that $\lambda=0$ in this case)

Case 2: $x=\lambda \to$ plug this in both $1^{st}$ and $3^{rd}$ equation

$$\begin{cases} y^2 = 2\lambda^2 \\ \lambda^2+y^2 = 1 \end{cases} \Rightarrow \begin{cases} y^2 = 2\lambda^2 \\ y^2 = 1-\lambda^2 \end{cases} \to 2\lambda^2 = 1-\lambda^2 \Rightarrow 3\lambda^2 = 1 \Rightarrow \lambda^2 = \frac{1}{3}$$

$x = \lambda = \pm\sqrt{\frac{1}{3}}$

$y^2 = 2\cdot\frac{1}{3} = \frac{2}{3} \Rightarrow y = \pm\sqrt{\frac{2}{3}}$

$\to$ 4 critical points $\boxed{P_1\left(-\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{2}{3}}\right)\ P_2\left(-\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{2}{3}}\right),\ P_3 = \left(\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{2}{3}}\right), P_4\left(\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{2}{3}}\right)}$

global min.

global max.

$f(P_1) = f(P_2) = -\sqrt{\frac{1}{3}}\cdot\left(\frac{2}{3}\right) < 0$

$f(P_3) = f(P_4) = \sqrt{\frac{1}{3}}\cdot\left(\frac{2}{3}\right) > 0$

**9.** (12 pts) Choose ONE proof. If you do both, only the larger score will be considered for this problem, but the second proof may give some bonus towards a previous problem where you may have lost some points. You can use the back of the page.

(A) Prove that for a differentiable function $f(x, y)$, the gradient is normal to the level curves of $f$.

(B) Show that if $z$ is implicitly defined as a function of $x$ and $y$ by $F(x, y, z) = 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}},$$

at all points where $\frac{\partial F}{\partial z} \neq 0$.

See textbook or class notes