NAME: Solution Key   Panther ID: ____________

Exam 3 - MAC 2313   Spring 2022

Important Rules:

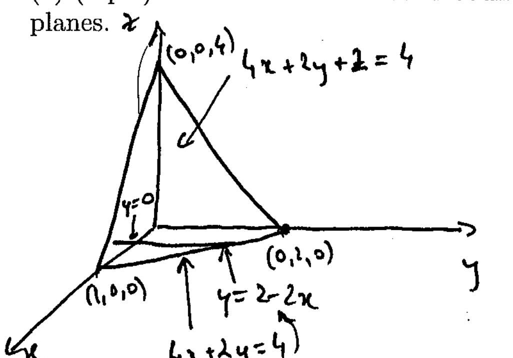
A. Any electronic device (cell phone, calculator of any kind, smart-watch, etc.) should be turned off at the beginning of the exam and placed in your bag, NOT in your pocket. Electronic items, notes, texts, or formula sheets should NOT be used at any time during the examination. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

Cheating attempts will lead to a score of zero on this exam, and possibly a report for academic misconduct.

B. Unless otherwise mentioned, to receive full credit you must show your work. Answers which are not supported by work might receive no credit. Solutions should be concise and clearly written. Incomprehensible work might not be considered.

1. (16 pts) Set up iterated double or triple integrals to represent each of the following. Don't spend time trying to evaluate the integrals. It is not required. (A sketch is required in each case.)

(a) (8 pts) The volume of the solid bounded in the first octant by the plane $4x + 2y + z = 4$ and the coordinate planes.



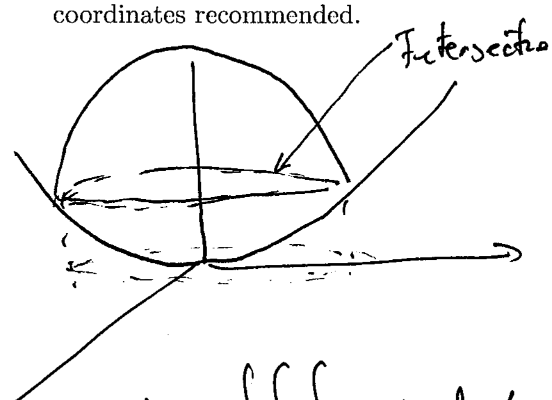
$$V = \iiint_D 1\, dV = \iint_R \left(\int_{z=0}^{z=4-4x-2y} 1\, dz \right) dA =$$

$$V = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=2-2x} \left(\int_{z=0}^{z=4-4x-2y} dz \right) dy \right) dx$$

$$\boxed{V = \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} \int_{z=0}^{z=4-4x-2y} dz\, dy\, dx}$$

← One of the possible good answers

(b) (8 pts) The volume of the solid bounded between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Cylindrical coordinates recommended.



$$\begin{cases} z = x^2 + y^2 \\ z = 8 - x^2 - y^2 \end{cases} \Rightarrow x^2 + y^2 = 8 - x^2 - y^2$$

$$\Rightarrow 2(x^2 + y^2) = 8 \Rightarrow$$

$$\Rightarrow \begin{cases} x^2 + y^2 = 4 \\ z = 4 \end{cases}$$ ← Intersection curve is a circle of radius 2 in the plane $z = 4$

$$V = \iiint_D 1\, dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r^2}^{z=8-r^2} 1\, dz\, r\, dr\, d\theta$$

2. (10 pts) Circle if each of the following statements is True or False. No justification is necessary.

(a) For any continuous function $f(x, y)$, $\int_1^3 \int_0^4 f(x, y)\, dx\, dy = \int_0^4 \int_1^3 f(x, y)\, dy\, dx$ **True** **False**

(b) The area of a region R in the plane is given by $A = \int_R \int 1\, dA$. **True** **False**

(c) For any function $\phi(x, y, z)$ with continuous second order partial derivatives, $\mathbf{Curl}(\nabla\phi) = \mathbf{0}$. **True** **False**

(d) Let R ~~be region~~ in the xy-plane bounded by $y = 4 - x^2$ and the x-axis. The centroid of R is on the x-axis.
True **False**

Centroid is on the y-axis (using the symmetry)

(e) For the change of variables, $(x = u - 2v,\ y = u + 2v)$, the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ equals 1. **True** **False**

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 - (-2) = 4 \neq 1$$

3. (10 pts) Use spherical coordinates to write an iterated integral that gives the mass of a spherical solid of radius a, if the density at each point is proportional to the distance from the center. Assume the constant of proportionality to be k. You DO NOT have to evaluate the integral.

Assume the sphere to have center at (0,0,0)

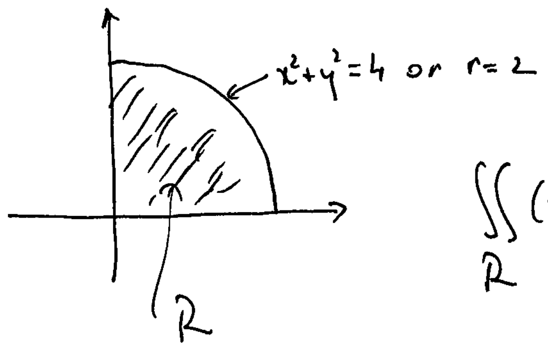
Density at each point $\delta = k \cdot \rho$

where $\rho = (x^2+y^2+z^2)^{\frac{1}{2}}$ = dist from the point to the origin.

$$M = \iiint_D \delta\, dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=a} k \cdot \rho \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

or $$\int_0^{2\pi} \int_0^{\pi} \int_0^{a} k \rho^3 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

4. (14 pts) Use polar coordinates to evaluate $\int\int_R (x^2+y^2)\, dA$ where R is the first quadrant portion of the disk $x^2+y^2 \leq 4$. Full computation is required for full credit.

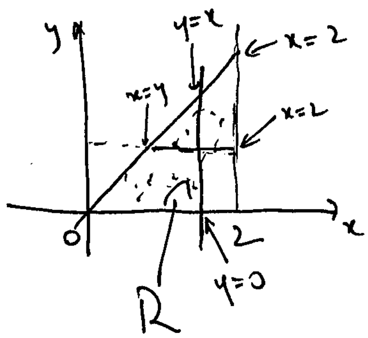


$$\iint_R (x^2+y^2)\,dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2} r^2\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2} r^3\,dr\,d\theta = \frac{\pi}{2}\left(\frac{r^4}{4}\Big|_{r=0}^{r=2}\right) = \frac{\pi}{2}\cdot\frac{16}{4} = \boxed{2\pi}$$

5. (14 pts) Evaluate the integral by reversing the order of integration $\int_0^2 \int_y^2 \cos(x^2)\,dx\,dy$. Full computation is required for full credit. Be sure to also sketch the region of integration.

$R = \{(x,y) \mid y \le x \le 2 \;,\; 0 \le y \le 2\}$



$$\int_0^2 \int_y^2 \cos(x^2)\,dx\,dy = \iint_R \cos(x^2)\,dA =$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=x} \cos(x^2)\,dy\,dx =$$

$$= \int_{x=0}^{x=2} y\cos(x^2)\Big|_{y=0}^{y=x} dx = \int_0^2 x\cos(x^2)\,dx =$$

$$= \frac{1}{2}\sin(x^2)\Big|_{x=0}^{x=2} = \boxed{\frac{\sin(4)}{2}}$$

guess & adjust
or sub $u = x^2$

...

6. (16 pts) (a) (12 pts) Show that the vector field $\mathbf{F}(x, y) = \overbrace{(\cos y + y\cos x)}^{f(x,y)}\mathbf{i} + \overbrace{(\sin x - x\sin y)}^{g(x,y)}\mathbf{j}$ is conservative and find a potential function.

Conservative v. field test $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$?

In our case $\dfrac{\partial g}{\partial x} = \cos x - \sin y = \dfrac{\partial f}{\partial y}$, so the vector field $\vec{F}$ is indeed conservative.

We look next for the potential function $\Phi(x,y)$ so that $\vec{F} = \nabla\Phi$

We must have
$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = \cos y + y\cos x \\ \dfrac{\partial \Phi}{\partial y} = \sin x - x\sin y \end{cases}$$

Integrating first relation with respect to x, we get
$$\Phi(x,y) = \int(\cos y + y\cos x)\,dx = x\cos y + y\sin x + c(y)$$

Differentiating this w.r.t. y, we check the second condition
$$\frac{\partial \Phi}{\partial y} = -x\sin y + \sin x + c'(y) = \sin x - x\sin y, \text{ so } c'(y) = 0$$
so $c(y) = k \leftarrow$ constant

Thus the potential function for this field is
$$\Phi(x,y) = x\cos y + y\sin x + k$$

(b) (4 pts) Using part (a), or by direct computation, find the work done by $\mathbf{F}(x, y)$ on a particle that moves along the line segment from $(0, 0)$ to $(\frac{\pi}{4}, \frac{\pi}{4})$.

Using the Fundam. Theorem for line integrals (of conservative v-fields

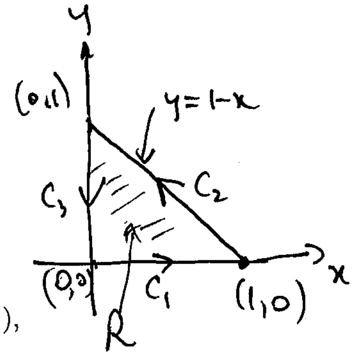
$$W = \Phi(P_{final}) - \Phi(P_{initial})$$

and this work is independent of the path

In our case,

$$W = \Phi\left(\frac{\pi}{4}, \frac{\pi}{4}\right) - \Phi(0,0) = \frac{\pi}{4}\cos\frac{\pi}{4} + \frac{\pi}{4}\sin\frac{\pi}{4} + \cancel{k} - \cancel{k}$$

$$W = \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} = \boxed{\frac{\pi\sqrt{2}}{4}}$$



7. (12 pts + 6 bonus pts) Compute the following line integral

$$\int_C 3xy\,dx + 2x^2\,dy\ ,$$

where C is the counter-clock-wise oriented triangle joining the points $(0,0)$, $(1,0)$, $(0,1)$, in any ONE of the following TWO ways:

(a) Using Green's Theorem.

(b) Using the definition of the line integral; that is, let $C = C_1 \bigcup C_2 \bigcup C_3$, parametrize each segment, etc.

Doing the problem correctly BOTH ways will give you 6 bonus points.

(a) Green's Theorem
$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

where R is a simply connected region with boundary the closed curve C oriented counter-clock wise

In our case,

$$\oint_C 3xy\,dx + 2x^2\,dy = \iint_R (4x - 3x)\,dA = \iint_R x\,dA =$$

See picture above

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x\,dy\,dx = \int_{x=0}^{x=1} x(1-x)\,dx = \int_{x=0}^{x=1} (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \boxed{\frac{1}{6}}$$

(b) without Green
$$\oint_C 3xy\,dx + 2x^2\,dy = \int_{C_1}(\ldots) + \int_{C_2}(\ldots) + \int_{C_3}(\ldots)$$

Parametrize each curve: C_1: $0 \le t \le 1$
$$\left.\begin{array}{l} x = t \Rightarrow dx = dt \\ y = 0 \Rightarrow dy = 0 \end{array}\right\} \Rightarrow \int_{C_1}(\ldots) = \int_0^1 0\cdot dt + 2t^2\cdot 0 = 0$$

C_2: $x = t$, $y = 1-t$ but t goes from $\underline{1 \text{ to } 0}$.
$dx = dt$ $dy = -dt$
$$\int_{C_2}(\ldots) = \int_1^0 3t(1-t)\,dt + 2t^2(-dt) = \ldots = \frac{1}{6}$$

C_3: $x = 0$, $y = t$ with t going again from 1 to 0
$dx = 0$ $dy = dt$
$$\int_{C_3}(\ldots) = \int_1^0 0 + 2\cdot 0\,dt = 0$$

Thus adding these $\int_C 3xy\,dx + 2x^2\,dy = 0 + \frac{1}{6} + 0 = \boxed{\frac{1}{6}}$

8. (14 pts) Choose ONE of the following. If you do BOTH, only the top score will count, but the second score may give some bonus toward a previous problem with a lower score.

(a) State and prove the fundamental theorem of line integrals (also known as the fundamental theorem of conservative vector fields).

(b) Compute the divergence of a 3D inverse-square field

$$\mathbf{F} = c\frac{\mathbf{r}}{|\mathbf{r}|^3} = c\Big(\frac{x}{\rho^3}\,\mathbf{i} + \frac{y}{\rho^3}\,\mathbf{j} + \frac{z}{\rho^3}\,\mathbf{k}\Big)\,,$$

where $|\mathbf{r}| = \rho = (x^2+y^2+z^2)^{1/2}$ and c is a constant.

(a) See class notes for (a) or textbook.

(b) $\operatorname{div}\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$, where $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$

To compute, start with $\rho = (x^2+y^2+z^2)^{\frac{1}{2}}$ and note that

$$\frac{\partial \rho}{\partial x} = \rho_x = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}}\cancel{2}x = \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{x}{\rho}$$

Similarly (using symmetry) $\rho_y = \frac{y}{\rho}$, $\rho_z = \frac{z}{\rho}$

$$\frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right) \underset{\substack{\uparrow \\ \text{quotient rule}}}{=} \frac{1\cdot\rho^3 - x\cdot 3\rho^2\cdot\rho_x}{\rho^6} = \frac{\rho^3 - 3x\rho^{\cancel{2}}\cdot\frac{x}{\cancel{\rho}}}{\rho^6} = \frac{\cancel{\rho}(\rho^2 - 3x^2)}{\cancel{\rho^6}\,\rho^5} = \frac{(x^2+y^2+z^2-3x^2)}{\rho^5}$$

$$\text{Thus } \frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right) = \frac{-2x^2+y^2+z^2}{\rho^5} \overset{\text{or}}{=} \frac{\rho^2-3x^2}{\rho^5}$$ ← maybe even better form

$$\text{Similarly } \frac{\partial}{\partial y}\left(\frac{y}{\rho^3}\right) = \ \cdots \ = \frac{\rho^2-3y^2}{\rho^5}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\rho^3}\right) = \ \cdots \ = \frac{\rho^2-3z^2}{\rho^5}$$

$$\text{Thus } \operatorname{div}(\vec{F}) = \frac{\rho^2-3x^2}{\rho^5} + \frac{\rho^2-3y^2}{\rho^5} + \frac{\rho^2-3z^2}{\rho^5} = \frac{3\rho^2 - 3(x^2+y^2+z^2)}{\rho^5} = \frac{3\rho^2-3\rho^2}{\rho^5}$$

$$\text{Thus } \operatorname{div}(\vec{F}) = 0$$

So a 3D-inverse square field is divergence free so incompressible (if it describes a fluid)

Note however that $\vec{F} = c\frac{\vec{r}}{|\vec{r}|^3}$ is NOT defined at $(0,0,0)$ so one should be careful about regions that contain the origin.