Name: Solution key   **Panther ID:** ____________

Exam 1   MAC 2313   Spring 2022

Important Rules:

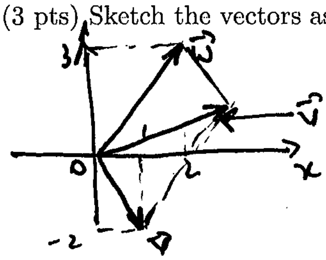
A. Any electronic device (cell phone, calculator of any kind, smart-watch, etc.) should be turned off at the beginning of the exam and placed in your bag, NOT in your pocket. Electronic items, notes, texts, or formula sheets should NOT be used at any time during the examination. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

Cheating attempts will lead to a score of zero on this exam, and possibly a report for academic misconduct.

B. Unless otherwise mentioned, to receive full credit you must show your work. Answers which are not supported by work might receive no credit. Solutions should be concise and clearly written. Incomprehensible work might not be considered.

1. (15 pts) Given the vectors $\mathbf{v} =< 1, -2 >$, $\mathbf{w} =< 2, 3 >$ in the plane, do the following:

(a) (3 pts) Sketch the vectors assuming that the initial point for both is the origin.



(b) (4 pts) Find the angle θ between $\mathbf{v}$ and $\mathbf{w}$ (answer as inverse trig. function ok).

$\vec{v}\cdot\vec{w} = |\vec{v}|\,|\vec{w}|\cos\theta$ so $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|}$

$\cos\theta = \dfrac{-4}{\sqrt{5}\cdot\sqrt{13}}$ so $\theta = \arccos\left(-\dfrac{4}{\sqrt{65}}\right)$ (it is an obtuse angle as the picture also suggests)

(c) (4 pts) Find a unit vector $\mathbf{u}$ in the direction opposite to the direction of $\mathbf{v}$.

$\vec{u} = -\dfrac{\vec{v}}{|\vec{v}|} = -\dfrac{1}{\sqrt{5}} < 1, -2 > = \dfrac{1}{\sqrt{5}} < -1, 2 >$

(d) (4 pts) If $\mathbf{v}$ and $\mathbf{w}$ represent forces, sketch the resultant force $\mathbf{v} + \mathbf{w}$ and determine its magnitude.

The sketch of $\vec{v}+\vec{w}$ is above

$\vec{v}+\vec{w} = \langle 3, 1\rangle$ so $|\vec{v}+\vec{w}| = |\langle 3, 1\rangle| = \boxed{\sqrt{10}}$

2. (15 pts) The following are multiple choice questions. Circle the correct answers. No justification needed, but some correct scratch work may give you a little partial credit in case of a wrong answer (3 pts each).

(i) Consider the sphere $x^2 + y^2 + z^2 - 2x = 4$. What is the radius of the sphere?

(A) 2 (B) $2\sqrt{2}$ (C) $\sqrt{5}$ (D) ~~$\sqrt{3}$~~

you should complete the square first!

(ii) A force $\mathbf{F} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ (in Newtons) moves an object along a line segment from $P(1,2,0)$ to $Q(3,4,2)$ (in meters). What is the work done by the force (in Joules)?

(A) 12 (B) 11 (C) 17 (D) 5

$W = \vec{F} \cdot \vec{PQ} = \langle 1,2,3\rangle \cdot \langle 2,2,2\rangle$

(iii) Determine whether the following planes are parallel, intersecting but not orthogonally, or intersecting orthogonally: $x + y - z = 0$ and $2x + 3y + z = 0$

(A) parallel (B) intersecting but not orthogonally (C) intersecting orthogonally

$\vec{n}_1 = \langle 1,1,-1\rangle$ $\vec{n}_2 = \langle 2,3,1\rangle$, so $\vec{n}_1 \not\parallel \vec{n}_2$ and $\vec{n}_1 \not\perp \vec{n}_2$ (as $\vec{n}_1 \cdot \vec{n}_2 \neq 0$)

(iv) Determine the type of the quadric surface given by $x^2 + 2y^2 - 3z = 0$.

(A) elliptic cone (B) elliptic paraboloid (C) ellipsoid (D) neither of the above

(v) If a particle moves with constant speed on a curve $\mathbf{r}(t)$, then its acceleration vector $\mathbf{a}(t)$

(A) must be the zero vector. (B) must have zero tangential component (C) must have zero normal component.

3. (12 pts) (a) (8 pts) Show that the line $x = 2 - t$, $y = 3 + 2t$, $z = t$ intersects the plane $x - 2y - z = 8$ and find the point of intersection.

Plugging in x, y, z from the parametric equations of the line into the equation of the plane, we get

$(2-t) - 2(3+2t) - t = 8$, so $2 - t - 6 - 4t - t = 8$

or $-6t = 12$ or $t = -2$

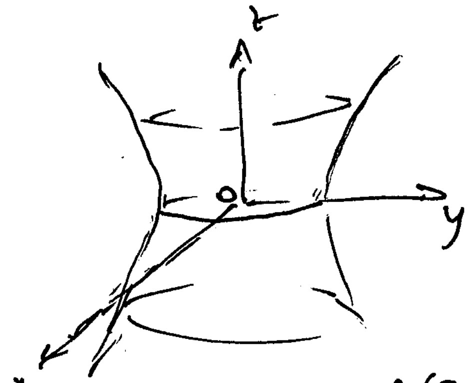
The point of intersection is $P(4,-1,-2)$ (by plugging in $t=-2$ back in the param. eqns. of the line)

(b) (4 pts) Is the line $x = 2 - t$, $y = 3 + 2t$, $z = t$ perpendicular to the plane $x - 2y - z = 8$? Briefly justify.

~~The~~ A directional vector for the line is $\vec{u} = \langle -1,2,1\rangle$, while a normal to the plane is $\vec{n} = \langle 1,-2,-1\rangle$

Note that $\vec{u} \parallel \vec{n}$ (since $\vec{n} = -\vec{u}$) so, yes, the line is perpendicular to the plane.

4. (12 pts) Consider the quadric surface $\frac{x^2}{25} + \frac{y^2}{9} - z^2 = 1$.



(a) (4 pts) Identify the surface by name and sketch its graph.

It is a hyperboloid with one sheet along the z-axis

(b) (4 pts) Find the x-axis intercepts of the surface (if any).

$\frac{x^2}{25} + 0 - 0 = 1$ so $x^2 = 25$, so $\boxed{x = \pm 5}$ ← The x-axis intercepts are $P_1(5,0,0)$ and $P_2(-5,0,0)$.

(c) (4 pts) Find the equation for the intersection of the surface with the plane $z = -1$. Is this intersection an ellipse, parabola, hyperbola, or neither one of these?

$\frac{x^2}{25} + \frac{y^2}{9} - (-1)^2 = 1$ so $\frac{x^2}{25} + \frac{y^2}{9} - 1 = 1$ or $\frac{x^2}{25} + \frac{y^2}{9} = 2$

The intersection is an ellipse.

5. (14 pts) Given the points $A(-1,-2,2)$, $B(1,-1,4)$, $C(2,-3,2)$ in 3-space:

(a) (7 pts) Find the parametric equations of the line through the points A and B.

Directional vector can be taken $\vec{v} = \vec{AB} = \langle 2, 1, 2 \rangle$

so $\begin{cases} x = -1 + 2t \\ y = -2 + t \\ z = 2 + 2t \end{cases}$, where I chose the initial point to be A

(b) (7 pts) Find an equation for the plane containing the points A, B, C.

Normal vector to the plane $\vec{n} = \vec{AB} \times \vec{AC}$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 3 & -1 & 0 \end{vmatrix} = +2\vec{i} - (-6)\vec{j} + (-5)\vec{k} = +2\vec{i} + 6\vec{j} - 5\vec{k}$$

Using point-normal equation of the plane (again using A as the point) we get

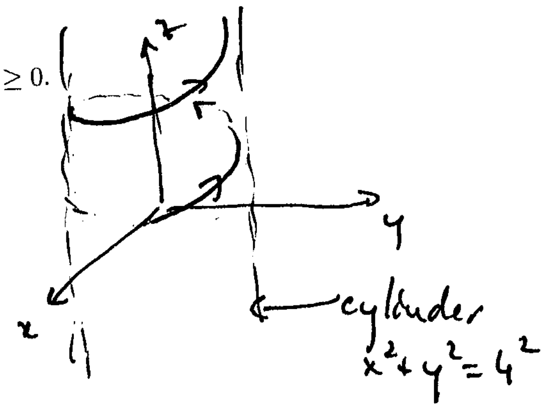
$$+2(x-(-1)) + 6(y-(-2)) - 5(z-2) = 0$$

or $+2(x+1) + 6(y+2) - 5(z-2) = 0$ (or equivalent forms)

6. (18 pts) Consider the curve $\mathbf{r}(t) =< 4\cos t\ ,\ 4\sin t\ ,\ 3t\ >$, for $t \geq 0$.

(a) (4 pts) Sketch a graph of $\mathbf{r}(t)$ in 3d and briefly describe it in words.

It's a cylindrical helix



(b) (7 pts) Find the length of the curve when $0 \leq t \leq 2\pi$.

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} 5\,dt = \boxed{10\pi}$$

$$\vec{r}\,'(t) = \langle -4\sin t,\ 4\cos t,\ 3 \rangle$$

$$\text{so } |\vec{r}\,'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = \sqrt{16+9} = \sqrt{25} = 5$$

(c) (7 pts) Find the curvature $\kappa(t)$ of the curve. Recall the best formula for computation of curvature is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\vec{r}\,''(t) = \langle -4\cos t,\ -4\sin t,\ 0 \rangle$$

$$\text{so } \vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4\sin t & 4\cos t & 3 \\ -4\cos t & -4\sin t & 0 \end{vmatrix} = 12\sin t\,\vec{i} - 12\cos t\,\vec{j} + 16\,\vec{k}$$

$$\kappa(t) = \frac{\sqrt{144\sin^2 t + 144\cos^2 t + 256}}{5^3} = \frac{\sqrt{144+256}}{125} = \frac{\sqrt{400}}{125} = \frac{20}{125} = \boxed{\frac{4}{25}}$$

7 (12 pts) The trajectory of a moving particle at time $t \geq 0$ is given by

$$\mathbf{r}(t) = 3\sin(2t)\,\mathbf{i} \;+\; 5\cos(2t)\,\mathbf{j} \;+\; 4\sin(2t)\,\mathbf{k}\,.$$

(a) (6 pts) Show that the trajectory lies on a sphere with the center at the origin and whose radius you need to find.

The equation of a sphere with center at $(0,0,0)$ is

$$x^2 + y^2 + z^2 = r^2$$

The curve $\vec{r}(t)$ in parametric form is $\{x = 3\sin(2t),\ y = 5\cos(2t),\ z = 4\sin(2t)\}$.
Replacing these in the equation ~~of sphere~~ above, we get

$$\left(3\sin(2t)\right)^2 + \left(5\cos(2t)\right)^2 + \left(4\sin(2t)\right)^2 = r^2 \quad \text{or}$$

$$9\sin^2(2t) + 25\cos^2(2t) + 16\sin^2(2t) = r^2 \quad \text{or}$$

$$25\left(\sin^2(2t) + \cos^2(2t)\right) = r^2 \quad \text{or } r^2 = 25 \quad \text{or } \underline{r = 5}.$$

Thus, the curve is on the sphere with center at the origin and of radius 5.

(b) (6 pts) Find the velocity vector $\mathbf{v}(t)$ and show that the speed is constant.

$$\vec{v}(t) = \vec{r}\,'(t) = 6\cos(2t)\,\vec{i} - 10\sin(2t)\,\vec{j} + 8\cos(2t)\,\vec{k}$$

(where the extra factor of 2 appears because of Chain Rule)

$$\text{speed} = |\vec{v}(t)| = \sqrt{6^2\cos^2(2t) + 10^2\sin^2(2t) + 8^2\cos^2(2t)} =$$

$$= \sqrt{36\cos^2(2t) + 64\cos^2(2t) + 100\sin^2(2t)} =$$

$$= \sqrt{100\left(\cos^2(2t) + \sin^2(2t)\right)} = \sqrt{100} = 10$$

Thus, the speed is constant $\underline{10}$.

8. (12 pts) Choose ONE proof. If you do BOTH, only the larger score will be considered for this problem, but the second score may give some bonus towards a previous problem where you may have lost points. You can use the back of the page, if needed.

(A) Find, with proof, the parametric equations for the ideal projectile motion. That is, assume there is no air friction, no wind, and that the only force acting on the projectile is the gravity. Assume also that the projectile is shot from a certain height y_0 with an initial speed $|\mathbf{v_0}|$ at an angle α with the horizontal.

For an additional **3 pts bonus** show also that if the projectile is shot from the ground (so $y_0 = 0$), then the horizontal distance the projectile travels until it hits the ground again (that is, its range) is given by

$$\text{range} = \frac{|\mathbf{v_0}|^2 \sin(2\alpha)}{g},$$

where g is the gravitational constant.

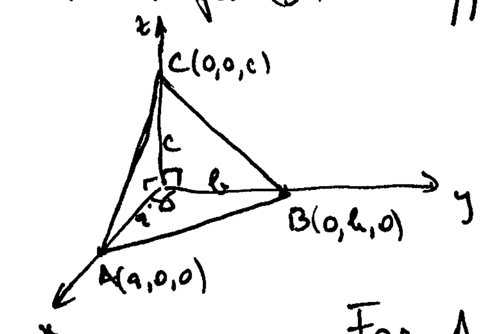
(B) Consider the usual 3d rectangular coordinate system with origin O and let $A(a,0,0)$ be an arbitrary point on the x-axis, $B(0,b,0)$ be an arbitrary point on the y-axis and $C(0,0,c)$ be an arbitrary point on the z-axis. Show the following 3d-version of Pythagorean Theorem

$$(Area(\triangle ABC))^2 = (Area(\triangle AOB))^2 + (Area(\triangle BOC))^2 + (Area(\triangle COA))^2$$

Hint: Use vectors to compute the area of triangle $\triangle ABC$. The areas of the triangles on the right side are easy to compute without vectors.

For (A) see the notes or the textbook.

Solution for (B) also appears in one of the lecture notes. Here it is again:



$\text{Area}(\triangle AOB) = \frac{a \cdot b}{2}$ (as it is a right angle triangle)

Similarly

$\text{Area}(\triangle BOC) = \frac{b c}{2}$ and $\text{Area}(\triangle COA) = \frac{c \cdot a}{2}$

For $\text{Area}(\triangle ABC)$ we use vectors to compute it (it's no longer a right angle triangle)

$$\text{Area}(\triangle ABC) = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\vec{i} + ac\vec{j} + ab\vec{k}$$

Thus $|\vec{AB} \times \vec{AC}| = \sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

Thus $(\text{Area}(\triangle ABC))^2 = \frac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2) = \left(\frac{ab}{2}\right)^2 + \left(\frac{b \cdot c}{2}\right)^2 + \left(\frac{c \cdot a}{2}\right)^2$

$= (\text{Area}(\triangle AOB))^2 + (\text{Area}(\triangle BOC))^2 + (\text{Area}(\triangle COA))^2$