Worksheet 10/26/2021 MAC 2283 Group nr. ___ **Names:** Solution Key

1. Circle if each of the following statements is true or false and then give a brief justification of your answer.

(a) The area of a region R in the xy-plane is given by $\int_R\int xy\,dA$ **True** **False**

Justification: $A = \iint_R 1\,dA$

(b) For any continuous functions $f(x), g(x)$ on $[a,b]$,

$\int_a^b f(x)\cdot g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\cdot\left(\int_a^b g(x)\,dx\right)$ **True** **False**

Justification: $\int_0^1 x\cdot x\,dx = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$, $\left(\int_0^1 x\,dx\right)\left(\int_0^1 x\,dx\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$\frac{1}{3} \neq \frac{1}{4}$

(c) If $f(x)$ is continuous on $[a,b]$, $g(y)$ continuous on $[c,d]$ and R is the rectangle $R = [a,b]\times[c,d]$, then

$\int_R\int f(x)\cdot g(y)\,dA = \left(\int_a^b f(x)\,dx\right)\cdot\left(\int_c^d g(y)\,dy\right)$ **True** **False**

Justification:

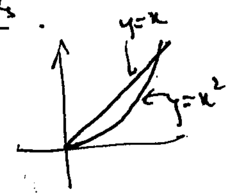
$$\iint_R f(x)g(y)\,dA = \int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x)g(y)\,dy\,dx = \int_a^b f(x)\left(\int_c^d g(y)\,dy\right)dx = \left(\int_c^d g(y)\,dy\right)\cdot\left(\int_a^b f(x)\,dx\right)$$

(d) For a continuous function $f(x,y)$, $\int_0^1\int_{x^2}^x f(x,y)\,dy\,dx = \int_{x^2}^x\int_0^1 f(x,y)\,dx\,dy$ **True** **False**

Justification: On the right side, the outer limits of integration must be constants.

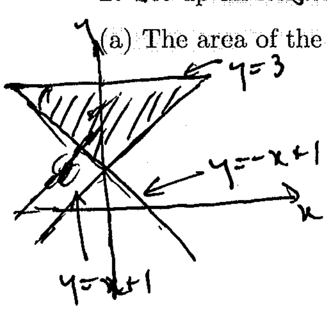
Correct change of order is, after drawing a picture

$$\int_0^1\int_{x^2}^x f(x,y)\,dy\,dx = \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} f(x,y)\,dx\,dy$$



2. Set up an iterated double integral to represent each of the following:

(a) The area of the region R, where R is the triangle bounded by $y = 3$, $y = -x+1$ and $y = x+1$.



$$A = \iint_R 1\,dA = \int_{y=1}^{y=3}\int_{x=1-y}^{x=y-1} dx\,dy$$ ← the easier order of integration

(b) The mass of a the thin plate covering the region R from part (a) and having density $\delta(x,y) = |xy|$.

$$M = \iint_R \delta\cdot dA \underset{\uparrow}{=} 2\iint_T |xy|\,dA$$

↑ because of symmetry of both the plate and density

where T is the triangle in the 1st quadrant only, so the absolute value can be dropped.

$$\boxed{M = 2\int_{y=1}^{y=3}\int_{x=0}^{y-1} xy\,dx\,dy}$$

3. (a) Evaluate the integral $\int\int_R \sin(y^3)\, dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$.
Hint: Choose the order of integration carefully.

(b) Evaluate the integral by first reversing the order of integration: $\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$

4. Use polar coordinates to find the volume of the solid bounded inside the cylinder $x^2 + y^2 = 9$ cut by the planes $z = 0$ and $z = 3 - x$.

5. Evaluate $\int\int_R \frac{1}{1 + x^2 + y^2}\, dA$,

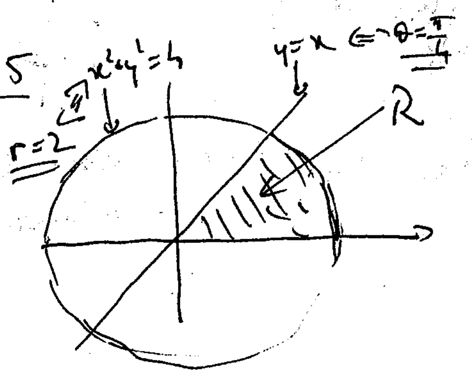
where R is the sector in the first quadrant bounded by $y = 0$, $y = x$, and $x^2 + y^2 = 4$.

6. (a) Use polar coordinates to find $\int\int_{\mathcal{R}} e^{-x^2-y^2}\, dA$ where $\mathcal{R}$ is the whole first quadrant of the xy-plane.

(b) Use part (a) to find the exact value of the improper integral $\int_0^{+\infty} e^{-x^2}\, dx$. This is an important integral in probability and statistics.

Pb. 6 $\rightarrow$ see class notes

Pb. 5



$$\iint_R \frac{1}{1+(x^2+y^2)}\, dA \overset{\text{polar coords}}{=}$$

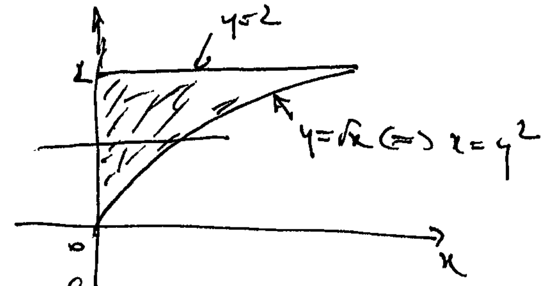
$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=2} \frac{1}{1+r^2}\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{1}{2} \ln(1+r^2)\Big|_{r=0}^{r=2} d\theta$$

$$= \frac{1}{2}(\ln 5 - \cancel{\ln 1}) \cdot \frac{\pi}{4} = \boxed{\frac{\pi \ln 5}{8}}$$

(3) Pb. 3 (a)

$$\iint_R \sin(y^3)\, dA \quad \text{, where } R \text{ is}$$



$$= \int_{y=0}^{y=2}\int_{x=0}^{x=y^2} \sin(y^3)\, dx\, dy$$

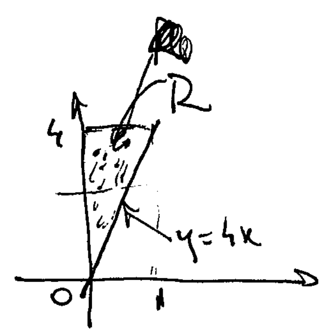
← only this order of integration makes the computation possible.

$$= \int_{y=0}^{y=2} y^2 \sin(y^3)\, dy \overset{u=y^3}{=} -\frac{1}{3}\cos(y^3)\Big|_{y=0}^{y=2}$$

$$= -\frac{1}{3}\cos(8) + \frac{1}{3}\cos(0) = \boxed{\frac{1}{3}(1-\cos 8)}$$

Pb 3 (b)

$$\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx = \iint_R e^{-y^2}\, dA$$



where $R = \{(x,y) \mid 4x \le y \le 4,\ 0 \le x \le 1\}$

Reversing order of integration, we get

$$\iint_R e^{-y^2}\, dA = \int_{y=0}^{y=4}\int_{x=0}^{x=\frac{y}{4}} e^{-y^2}\, dx\, dy =$$

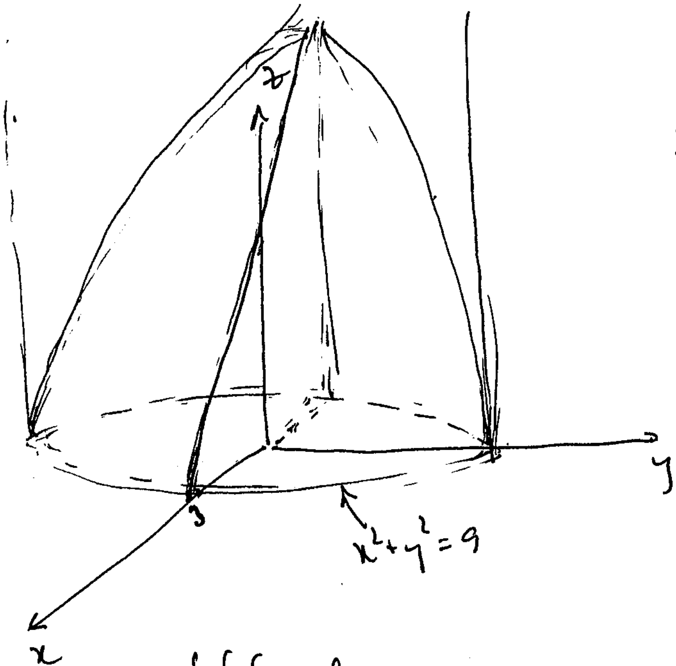
$$= \int_{y=0}^{y=4} \frac{y}{4} e^{-y^2}\, dy = -\frac{1}{8}\left(e^{-y^2}\right)\Big|_{y=0}^{y=4} = \frac{1}{8}\left(1-e^{-16}\right)$$

↑ $u=-y^2$, etc or guess & adjust

$\approx \frac{1}{8}$

↑ as e^{-16} is very small

Pb. 4.



Picture is a bit difficult to draw for this one, but as you integrate first with respect to z, the limits of integration for z will be $z=0$ (the xy-plane) and $z=3-x$ (the slanted plane cutting the cylinder)

$$V = \iiint_G 1\, dV$$

$$V = \iint_R \left(\int_{z=0}^{z=3-x} 1\, dz \right) dA = \iint_R (3-x)\, dA$$

where now R is the disk $x^2+y^2 \le 9$ in the xy-plane

Switching to polar coordinates,

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (3 - r\cos\theta)\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (3r - r^2\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{3r^2}{2} \Big|_{r=0}^{r=3} - \frac{r^3}{3}\cos\theta \Big|_{r=0}^{r=3} \right) d\theta = \int_0^{2\pi} \left(\frac{27}{2} - 9\cos\theta \right) d\theta$$

$$= \frac{27}{2}\cdot 2\pi - 9\sin\theta \Big|_{\theta=0}^{\theta=2\pi} = \boxed{27\pi}$$