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Worksheet 11/02 MAC 2283 – Fall'21 **Group nr:** ___ **Names:** Solution Key

1. Find the average temperature in the box $D = \{(x, y, z) \,|\, 0 \le x \le \ln 2,\ 0 \le y \le \ln 4,\ 0 \le z \le \ln 8\ \}$ with a temperature distribution $T(x, y, z) = 128e^{-x-y-z}$.

2. (a) Let G be the solid tetrahedron in the first octant bounded by the coordinate planes and the plane $3x + 2y + z = 6$. Find the volume of G using an iterated triple integral of the form $\int_?^? \int_?^? \int_?^? \,dz\,dx\,dy$.

(b) Now, set up an iterated triple integral that gives the volume of G, but this time the order of integration should be $\int_?^? \int_?^? \int_?^? \,dy\,dx\,dz$. You don't have to compute the integral this time.

(c) If the density of the solid G from part (a) is $\delta(x, y, z) = z$, set up an iterated triple integral that gives the mass of the solid. You can choose your favorite order of integration this time, and again, you don't have to compute the integral.

3. Use cylindrical coordinates (r, θ, z), to find the volume of the solid enclosed between the ellipsoid $4x^2 + 4y^2 + z^2 = 5$ and the elliptical (actually circular) paraboloid $z = x^2 + y^2$.

4. Find the mass of a spherical ball of radius a if the density is inversely proportional to the square of the distance from the center. That is, the density function is given by $\delta = \frac{k}{\rho^2}$, where k is a constant of proportionality. Use spherical coordinates.

Pb. 1.

$$\text{Avg. Temp.} = \frac{\iiint_D T(x,y,z)\,dV}{\text{Vol}(D)}$$

$\text{Vol}(D) = (\ln 2)\cdot(\ln 4)\cdot(\ln 8)$ (as it is a box with those dimensions)

$$\iiint_D T(x,y,z)\,dV = \int_0^{\ln 8}\int_0^{\ln 4}\int_0^{\ln 2} 128\, e^{-x-y-z}\,dx\,dy\,dz \stackrel{\leftarrow}{=} \qquad e^{-x-y-z} = e^{-x}\cdot e^{-y}\cdot e^{-z}$$

$$= 128\left(\int_0^{\ln 2} e^{-x}\,dx\right)\left(\int_0^{\ln 4} e^{-y}\,dy\right)\left(\int_0^{\ln 8} e^{-z}\,dz\right) =$$

$$= 128\left(1 - e^{-\ln 2}\right)\left(1 - e^{-\ln 4}\right)\left(1 - e^{-\ln 8}\right) = 128\left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{4}\right)\left(1-\tfrac{1}{8}\right)$$

$$\text{Avg. Temp} = \frac{128\left(\frac{1}{2}\right)\cdot\frac{3}{4}\cdot\frac{7}{8}}{(\ln 2)(\ln 4)(\ln 8)} \approx \boxed{21.02^\circ}$$

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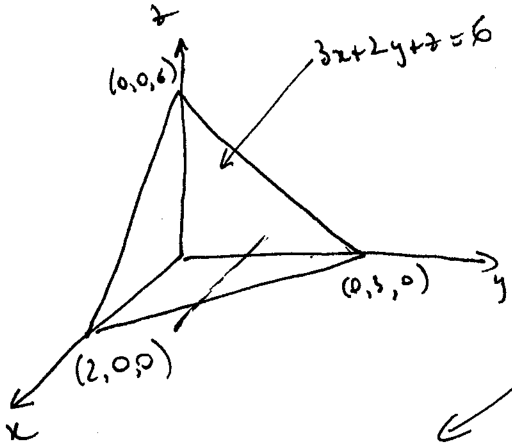
Pb. 2



(a) $V = \iiint_G 1 \, dV$

$$\boxed{V = \int_{y=0}^{y=3} \int_{x=0}^{x=2-\frac{2y}{3}} \int_{z=0}^{z=6-3x-2y} dz \, dx \, dy}$$

I will let you compute

(b)

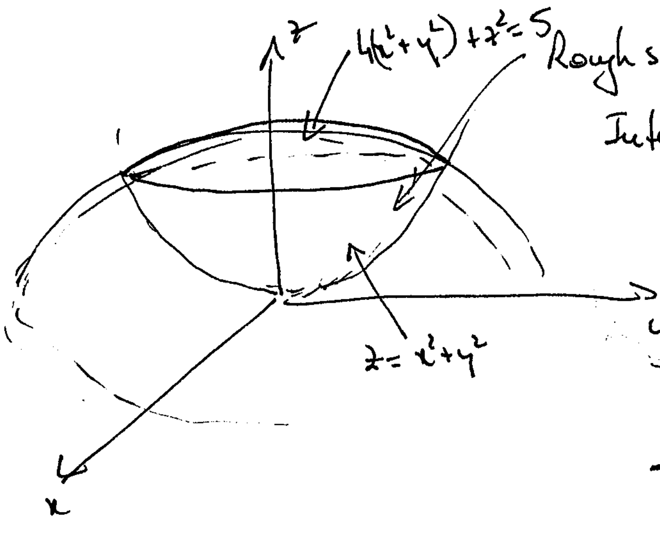
$$\boxed{V = \int_{z=0}^{z=6} \int_{x=0}^{x=\frac{6-z}{3}} \int_{y=0}^{y=\frac{6-3x-z}{2}} dy \, dx \, dz}$$

(c) Using the order from (a)

$$\boxed{M = \int_{y=0}^{y=3} \int_{x=0}^{x=2-\frac{2y}{3}} \int_{z=0}^{z=6-3x-2y} z \, dz \, dx \, dy}$$

density

Pb. 3



Rough sketch of the solid

Intersection: $\begin{cases} 4(x^2+y^2) + z^2 = 5 \\ z = x^2+y^2 \end{cases} \Rightarrow$

$\Rightarrow 4z + z^2 = 5 \Rightarrow$

$\Rightarrow z^2 + 4z - 5 = 0 \Rightarrow (z+5)(z-1) = 0$

$\Rightarrow \cancel{z = -5}, \ \boxed{z = 1}$

cannot be a solution as $z = x^2+y^2 \geq 0$

Thus, the intersection curve is the circle

$x^2+y^2 = 1$ in the plane $z = 1$

③ Continuation for solution of Pb. 3.

$$V = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r^2}^{z=\sqrt{5-4r^2}} 1 \, dz \, r \, dr \, d\theta$$

The set up is worth most of the points, but I'll also do the computation for this one

$$V = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} \left(\sqrt{5-4r^2} - r^2\right) r \, dr \, d\theta$$

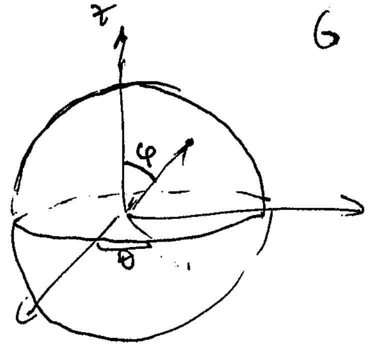
$$V = 2\pi \int_{r=0}^{r=1} \left(r\sqrt{5-4r^2} - r^3\right) dr$$

guess & adjust

$$V = 2\pi \left[-\frac{1}{12}\left(5-4r^2\right)^{\frac{3}{2}}\Big|_{r=0}^{r=1} - \frac{r^4}{4}\Big|_{r=0}^{r=1} \right]$$

$$V = 2\pi \left[-\frac{1}{12}(1)^{\frac{3}{2}} + \frac{1}{12}(5)^{\frac{3}{2}} - \frac{1}{4} \right] = \boxed{2\pi \left[\frac{5\sqrt{5}}{12} - \frac{1}{3} \right]}$$

Pb. 4.



$$M = \iiint_G \delta \, dV = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=a} \frac{k}{\rho^2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi} ka \sin\varphi \, d\varphi \, d\theta = ka(2\pi) \cdot (-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi}$$

$$= \boxed{4\pi k a}$$