(1) Solution key for Worksheet 11/09

Pbs. 1 & 2 were solved in class. Please see the class notes

Pb. 3 is an exercise that you should do.

Solution for Pb. 4

(a)

$$\begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \Rightarrow \text{Jacobian } \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

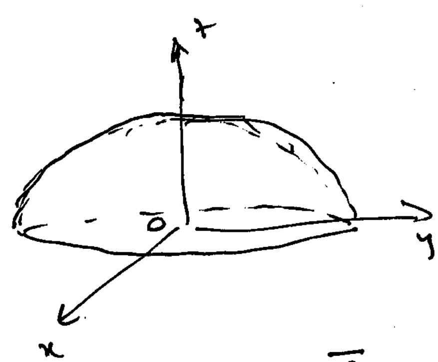
As mentioned in the problem, the transformation maps the solid sphere $G: u^2+v^2+w^2 \le 1$ onto the solid ellipsoid $E: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$, thus

$$V(E) = \iiint_E 1 \, dV_{x,y,z} \underset{\substack{\text{change of} \\ \text{variable}}}{=} \iiint_G \underbrace{abc}_{\text{Jacobian}} \, dV_{u,v,w}$$

For the integral on the right now use spherical coords., or directly just use the known formula for the volume of a sphere

Thus $V(E) = abc \cdot \frac{4\pi \cdot 1^3}{3} = \boxed{\frac{4\pi}{3} abc}$

(b)



Because of symmetry, the centroid of the upper-half ellipsoid will be on the z-axis.

Thus $\bar{x} = 0$, $\bar{y} = 0$.

$$\bar{z} = \frac{\iiint_{\tilde{E}} z \, dV}{\mathrm{vol}(\tilde{E})},$$ where $\tilde{E}$ is the upper-half of the ellipsoid

Thus $\mathrm{vol}(\tilde{E}) = \frac{1}{2}\mathrm{vol}(E) = \frac{2\pi}{3} abc$

(2)

$$\iiint_{\tilde{E}} z\, dV_{x,y,z} \underset{\substack{\uparrow \\ \text{change of} \\ \text{variables} \\ \text{from (a)}}}{=} \iiint_{\tilde{G}} cw \cdot abc\, dV_{u,v,w} = abc^2 \iiint_{\tilde{G}} w\, dV_{u,v,w} =$$

$$\underset{\substack{\uparrow \\ \text{use spherical} \\ \text{coords.}}}{=} abc^2 \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=1} \rho\cos\varphi \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$\sin(\varphi)\cos(\varphi) = \frac{1}{2}\sin(2\varphi)$

$$= \frac{abc^2}{2} \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=1} \rho^3 \sin(2\varphi)\, d\rho\, d\varphi\, d\theta =$$

$\int_0^1 \rho^3 d\rho = \frac{\rho^4}{4}\Big|_0^1 = \frac{1}{4}$

$$= \frac{abc^2}{2} \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \frac{1}{4}\sin(2\varphi)\, d\varphi\, d\theta =$$

$\int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \sin(2\varphi)\, d\varphi = -\frac{1}{2}\cos(2\varphi)\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} = -\frac{1}{2}(-2)$

$$= \frac{abc^2}{8}\int_{\theta=0}^{\theta=2\pi} 1 \cdot d\theta = \frac{abc^2}{8}\cdot 2\pi = \frac{abc^2 \cdot \pi}{4}$$

Thus $\bar{z} = \dfrac{\frac{\pi abc^2}{4}}{\frac{2\pi abc}{3}} = \boxed{\dfrac{3c}{8}}$