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Worksheet - REVISED - 11/23 - MAC 2283 Fall 2021 **NAMES:** Solution Key

This is a homework due Tuesday, Nov. 30.

1. Evaluate the line integral $\int_C (x+2y)\,dx + (x-y)\,dy$ along the curve $C: x = \cos t,\ y = 2\sin t,\ 0 \le t \le \pi/4$.

2. (a) Find the work done by the force field $F(x,y) = (x^2+xy)\,\mathbf{i} + (y - x^2y)\,\mathbf{j}$ on a particle that moves along the curve $C: x = t,\ y = 1/t,\ 1 \le t \le 3$.

(b) Is the vector field in part (a) conservative? Justify your answer.

3. (a) Verify that the vector field $F(x,y) = 2xy^3\,\mathbf{i} + (1+3x^2y^2)\,\mathbf{j}$ is conservative in the whole plane and find a potential function ϕ.

(b) Use the potential you found in part (a) to find the work of the field $\mathbf{F}$ along any smooth path from the point $(1,-1)$ to the point $(1,1)$.

4. (a) Show that if $\mathbf{F}(x,y,z)$ is a 3d conservative vector field, then $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$. *Hint:* You just have to show that $\mathbf{curl}\,\nabla\phi = \mathbf{0}$.

Note: As stated in class, the converse of this implication is also true if the vector field is defined on a simply connected region (that is, a region without holes): if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is conservative. (You don't have to prove this implication.)

(b) Compute $\mathbf{curl}\,\mathbf{F}$, for a 2-d vector field $\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$, and note that $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ if and only if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Thus, again the test for conservative vector fields in 2d can be restated as $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$.

Hint: Consider $F(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j} + 0\,\mathbf{k}$

Solution Pb. 1.: $x = \cos t$, $y = 2\sin t$ $\quad 0 \le t \le \frac{\pi}{4}$ (it is an arc of an ellipse)

$dx = -\sin t\,dt$, $dy = 2\cos t\,dt$

so $\displaystyle\int_C (x+2y)\,dx + (x-y)\,dy = \int_{t=0}^{t=\frac{\pi}{4}} (\cos t + 4\sin t)(-\sin t)\,dt + (\cos t - 2\sin t)(2\cos t)\,dt$

$$= \int_0^{\frac{\pi}{4}} \left(-4\sin^2 t - \sin t\cos t + 2\cos^2 t - 4\sin t\cos t\right) dt =$$

$$= \int_0^{\frac{\pi}{4}} \left(-4\sin^2 t + 2\cos^2 t - 5\sin t\cos t\right) dt =$$ ← use double angle identities

$$= \int_0^{\frac{\pi}{4}} \left(-4\cdot\frac{1-\cos(2t)}{2} + 2\cdot\frac{1+\cos(2t)}{2} - \frac{5}{2}\sin(2t)\right) dt$$

$$= \int_0^{\frac{\pi}{4}} \left(-1 + 3\cos(2t) - \frac{5}{2}\sin(2t)\right) dt =$$

$$= \left(-t + \frac{3}{2}\sin(2t) + \frac{5}{4}\cos(2t)\right)\Big|_{t=0}^{t=\frac{\pi}{4}} = -\frac{\pi}{4} + \frac{3}{2} - \frac{5}{4} = \boxed{\frac{1-\pi}{4}}$$

②

Solution Pb. 2.

(a) $W = \int_C \vec{F} \cdot d\vec{r} = \int_C (x^2+xy)\,dx + (y-x^2y)\,dy$

$C: \quad x = t \,, \quad y = \frac{1}{t} \qquad 1 \le t \le 3$

$dx = dt \quad dy = -\frac{1}{t^2}\,dt$

$$W = \int_{t=1}^{t=3} \left(t^2 + t\cdot\frac{1}{t}\right)dt + \left(\frac{1}{t} - t^2\cdot\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)dt$$

$$W = \int_{t=1}^{t=3} \left(t^2 + 1 - \frac{1}{t^3} + \frac{1}{t}\right)dt = \left(\frac{t^3}{3} + t - \left(-\frac{1}{2}\right)t^{-2} + \ln t\right)\Bigg|_{t=1}^{t=3}$$

$$W = \left(\frac{3^3}{3} + 3 + \frac{1}{2\cdot 3^2} + \ln 3\right) - \left(\frac{1}{3} + 1 + \frac{1}{2\cdot 1} + \cancel{\ln 1}\right) = \ldots$$

(b) The vector field is <u>not</u> conservative, as it fails the test

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} \qquad \frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(y - x^2y) = -2xy$$

$$\not\parallel$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy) = x$$

Solution Pb. 3

(a) $\vec{F}(x,y) = 2xy^3\,\vec{i} + (1+3x^2y^2)\,\vec{j}$ is conservative, as

$$\frac{\partial g}{\partial x} = 6xy^2 = \frac{\partial f}{\partial y}$$

Potential Φ: $\begin{cases} \dfrac{\partial \Phi}{\partial x} = 2xy^3 \\ \dfrac{\partial \Phi}{\partial y} = (1+3x^2y^2) \end{cases}$ $\Rightarrow \Phi(x,y) = \int 2xy^3\,dx = x^2y^3 + c(y)$

$$\frac{\partial \Phi}{\partial y} = 3x^2y^2 + c'(y) = 1 + 3x^2y^2$$

$$\Rightarrow c'(y) = 1 \Rightarrow c(y) = y + \text{const}$$

Thus $\Phi(x,y) = x^2y^3 + y + \text{const}$ is a potential function

(b) $W = \Phi(1,1) - \Phi(1,-1)$

$W = 2 - (-2) = 4$

(3)

Solution Pb. 4

(a) If $\vec{F}$ is conservative, then $\vec{F} = \nabla\Phi = \frac{\partial\Phi}{\partial x}\vec{\imath} + \frac{\partial\Phi}{\partial y}\vec{\jmath} + \frac{\partial\Phi}{\partial z}\vec{k}$

or $\vec{F} = \langle \Phi_x, \Phi_y, \Phi_z \rangle$

$$\text{curl}\,\vec{F} = \text{curl}(\nabla\Phi) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \Phi_x & \Phi_y & \Phi_z \end{vmatrix} = \vec{\imath}\left(\Phi_{yz} - \Phi_{zy}\right) - \vec{\jmath}\left(\Phi_{zx} - \Phi_{xz}\right)$$

$$+ \vec{k}\left(\Phi_{xy} - \Phi_{yx}\right) = \vec{0}$$

as mixed partial derivatives commute.

(b) $\vec{F}(x,y) = f(x,y)\vec{\imath} + g(x,y)\vec{\jmath} + 0\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & 0 \end{vmatrix} = \underbrace{\vec{\imath}\cdot 0}_{\text{as } \frac{\partial g}{\partial z} = 0} - \underbrace{\vec{\jmath}\cdot 0}_{\text{as } \frac{\partial f}{\partial z} = 0} + \vec{k}\left(g_x - f_y\right)$$

Thus, $\text{curl}\,\vec{F} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$, so $\text{curl}\,\vec{F} = \vec{0} \iff \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$