Worksheet 11/30 - MAC 2283 Fall 2021 NAMES: ____________________

1. **Green's theorem in circulation form**: Let R be a simply connected region in the plane (that is, R has no holes) whose boundary is a simple closed curve C, piecewise smooth and oriented counterclockwise. Let $\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$ be a vector field defined on a larger open set containing R that models a fluid flow. The *circulation* of $\mathbf{F}$ along C is defined as the line integral

$$\text{Circulation} = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \oint_C \mathbf{F}\cdot dr = \oint_C f\,dx + g\,dy \ .$$

Applying Green's Theorem,

$$\oint_C f\,dx + g\,dy = \int\!\!\int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \ .$$

With the exercise that you did in your previous take home quiz, the integrand on the right side is related to the **curl F**. More precisely,

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) = \mathbf{curl\,F}\cdot\mathbf{k}\ .$$

Thus, the **Green's theorem in circulation form** is

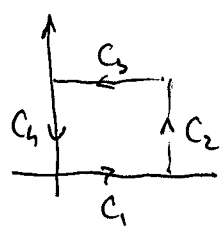
$$\text{Circulation} = \oint_C \mathbf{F}\cdot dr = \int\!\!\int_R (\mathbf{curl\,F}\cdot\mathbf{k})\,dA = \text{Integral over } R \text{ of the normal component of the curl}\ .$$

(a) Use the line integral in the definition to find the circulation of the vector field $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ on the square C cut from the the first quadrant by the lines $x = 1$ and $y = 1$.

(b) Use Green's theorem to find the circulation of the same vector field $\mathbf{F}$ as in (a) using this time the double integral in the formula.

(c) What is the circulation of a *conservative* vector field $\mathbf{F} = \nabla\phi$ over a simple closed curve C that bounds a simply connected region R? Briefly justify.

Solution: (a)



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

C_1: $x = t$, $y = 0 \Rightarrow dx = dt$ $dy = 0$ $\quad 0 \le t \le 1$

C_2: $x = 1$, $y = t \Rightarrow dx = 0$, $dy = dt$ $\quad 0 \le t \le 1$

C_3: $x = t$, $y = 1 \Rightarrow dx = dt$, $dy = 0$
but $1 \ge t \ge 0$ (here t goes from 1 to 0)

C_4: $x = 0$, $y = t \Rightarrow dx = 0$, $dy = dt$ $\quad 1 \ge t \ge 0$

$\vec{F}\cdot d\vec{r} = xy\,dx + y^2 dy$

$$\text{Circulation} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} + \oint_{C_3} + \oint_{C_4} =$$

$$= \int_0^1 0\,dt + \cancel{\int_0^1 t^2 dt} + \int_1^0 t\,dt + \cancel{\int_1^0 t^2 dt}$$

$$\text{Circulation} = -\int_0^1 t\,dt = \boxed{-\frac{1}{2}}$$

(b) With Green, $\text{Circulation} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA =$

$$= \iint_R (0 - x)\,dA = -\int_0^1\int_0^1 x\,dy\,dx = \ldots = \boxed{-\frac{1}{2}}$$

(c) If $\vec{F} = \vec{\nabla}\Phi = \Phi_x \vec{i} + \Phi_y \vec{j}$

$\text{curl}(\vec{F}) = \text{curl}(\vec{\nabla}\Phi) = \vec{0}$

Thus, $\text{Circulation} = \iint_G (\text{curl}\,\vec{F})\cdot\vec{k}\,dA = \boxed{0}$ ← for a conservative v.f.

2. Green's theorem in flux form: Let R be a simply connected region in the plane (that is, R has no holes) whose boundary is a simple closed curve C, piecewise smooth and oriented counterclockwise. Again, let $\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}$ be a vector field defined on a larger open set containing R that models a fluid flow. The *outward flux* of $\mathbf{F}$ along C is defined as the line integral

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C f\, dy - g\, dx \ .$$

Applying Green's Theorem,

$$\oint_C -g\, dx + f\, dy = \int\int_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA \ .$$

But the integrand on the right is the divergence of the vector field $\mathbf{F}$. Thus, **Green's theorem in flux form** is

$$\text{Flux} = \oint_C f\, dy - g\, dx = \int\int_R (\operatorname{div} \mathbf{F})\, dA = \text{Integral over } R \text{ of the divergence of } \mathbf{F} \ .$$

(a) Use the line integral in the definition to find the flux of the vector field $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ on the square C cut from the the first quadrant by the lines $x = 1$ and $y = 1$.

(b) Use Green's theorem to find the flux of the same vector field using the double integral in the formula.

(c) Apply Green's theorem to find an expression for the flux of a *conservative* vector field $\mathbf{F} = \nabla\phi$ over a simple closed curve C that bounds a simply connected region R.

Solution: (a) As before, $C = C_1 \cup C_2 \cup C_3 \cup C_4$, with the same parametrizations chosen in Pb. 1

In this case, $\vec{F}\cdot\vec{n} = f\,dy - g\,dx = xy\,dy - y^2\,dx$

$$\oint_{C_1} \vec{F}\cdot\vec{n}\, ds = \int_0^1 0\, dt = 0, \quad \oint_{C_2} \vec{F}\cdot\vec{n}\, ds = \int_0^1 t\, dt = \frac{1}{2}$$

$$\oint_{C_3} \vec{F}\cdot\vec{n}\, ds = \int_1^0 (-1)\, dt = \boxed{1} \quad \oint_{C_4} \vec{F}\cdot\vec{n}\, ds = \int_1^0 0 = 0$$

$$\Rightarrow \text{Flux} = \int_C \vec{F}\cdot\vec{n}\, ds = 1 + \frac{1}{2} = \frac{3}{2}$$

(b) With Green,

$$\text{Flux} = \oint_C (f\,dy - g\,dx) = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \iint_R (y + 2y)\, dA = \int_0^1\int_0^1 3y\, dy\, dx = \boxed{\frac{3}{2}}$$

(c) If $\vec{F} = \vec{\nabla}\Phi = \Phi_x \vec{i} + \Phi_y \vec{j}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(\Phi_x) + \frac{\partial}{\partial y}(\Phi_y) = \Phi_{xx} + \Phi_{yy} = \Delta\Phi$$

Laplacian of Φ

$$\text{Thus Flux} = \oint_C (f\,dy - g\,dx) = \iint_R (\operatorname{div}\vec{F})\, dA \underset{\text{if } \vec{F} = \nabla\Phi}{=} \boxed{\iint_R (\Delta\Phi)\, dA}$$