Worksheet 4/12 - MAC 2313 Spring 2022 **NAMES:** ____________________

1. (a) Compute $\mathbf{curl}\,\mathbf{F}$, for a 2-d vector field $\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$. Note that $\mathbf{curl}\,\mathbf{F} = \mathbf{0} \Leftrightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

(b) Show that if $\mathbf{F}(x,y,z)$ is a conservative vector field on a certain region R in $\mathbf{R}^3$, then $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ on R. *Hint:* You just have to show that $\mathbf{curl}\,\nabla\phi = \mathbf{0}$. (Assume that the potential function ϕ has continuous second order partial derivatives everywhere on R.)

Note: As stated in class, the converse of the implication in (b) is also true (but a bit harder to prove) when the vector field is defined on a connected AND simply connected region R (that is, a connected region without holes). Thus, here is the statement of the **Test for Conservative Vector Fields Theorem:**

On a connected and simply connected region R, $\mathbf{F}$ is conservative if and only if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is conservative. If $\mathbf{F}$ is 2-dimensional, that is, $\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$, $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ if and only if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

2. Green's theorem in circulation form: Let R be a simply connected region in the plane (that is, R has no holes) whose boundary is a simple closed curve C, piecewise smooth and oriented counterclockwise. Let $\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$ be a vector field defined on a larger open set containing R that models a fluid flow. The *circulation* of $\mathbf{F}$ along C is defined as the line integral

$$\text{Circulation} = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C f\,dx + g\,dy \ .$$

Applying Green's Theorem,

$$\oint_C f\,dx + g\,dy = \int\int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \ .$$

With the problem 1(a) above, the integrand on the right side is related to the $\mathbf{curl}\,\mathbf{F}$. More precisely,

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) = \mathbf{curl}\,\mathbf{F}\cdot\mathbf{k} \ .$$

Thus, the **Green's theorem in circulation form** is

$$\text{Circulation} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R (\mathbf{curl}\,\mathbf{F}\cdot\mathbf{k})\,dA = \text{ Integral over } R \text{ of the normal component of the curl} \ .$$

(a) Use the line integral in the definition to find the circulation of the vector field $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ on the square $\mathcal{C}$ cut from the the first quadrant by the lines $x = 1$ and $y = 1$.

(b) Use Green's theorem to find the circulation of the same vector field $\mathbf{F}$ as in (a) using this time the double integral in the formula.

(c) What is the circulation of a *conservative* vector field $\mathbf{F} = \nabla\phi$ over a simple closed curve C that bounds a simply connected region R? Briefly justify.

3. Green's theorem in flux form: Let R be a simply connected region in the plane (that is, R has no holes) whose boundary is a simple closed curve C, piecewise smooth and oriented counterclockwise. Again, let $\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}$ be a vector field defined on a larger open set containing R that models a fluid flow. The *outward flux* of $\mathbf{F}$ along C is defined as the line integral

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C f\, dy \; - \; g\, dx \;\; .$$

Applying Green's Theorem,

$$\oint_C -g\, dx \; + \; f\, dy = \int\int_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dA \;\; .$$

But the integrand on the right is the divergence of the vector field $\mathbf{F}$. Thus, **Green's theorem in flux form** is

$$\text{Flux} = \oint_C f\, dy \; - \; g\, dx = \int\int_R (\text{div}\, \mathbf{F})\, dA = \text{ Integral over } R \text{ of the divergence of } \mathbf{F} \;.$$

(a) Use the line integral in the definition to find the flux of the vector field $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ on the square $\mathcal{C}$ cut from the the first quadrant by the lines $x = 1$ and $y = 1$.

(b) Use Green's theorem to find the flux of the same vector field using the double integral in the formula.

(c) Apply Green's theorem to find an expression for the flux of a *conservative* vector field $\mathbf{F} = \nabla\phi$ over a simple closed curve C that bounds a simply connected region R.