Worksheet 02/24 - MAC 2313 **Group nr.** ____ **NAMES:** Solution Key

1. Consider the function $f(x, y) = \ln(1 + 2x + 3y)$. $f(0,0) = \ln(1) = 0$

(a) Find the linearization $L(x, y)$ (or the local linear approximation) of $f(x, y)$ at $(0, 0)$.

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$, where in our case $(x_0,y_0) = (0,0)$

$f_x = \frac{2}{1+2x+3y} \Rightarrow f_x(0,0) = \frac{2}{1} = 2$

$f_y = \frac{3}{1+2x+3y} \Rightarrow f_y(0,0) = \frac{3}{1} = 3$

$\Rightarrow L(x,y) = 0 + 2(x-0) + 3(y-0) = 2x+3y$ ← linearization

or $\ln(1+2x+3y) \approx 2x+3y$ ← local lin. approximation

(b) What is the equation of the tangent plane to the graph of $f(x, y)$ at $(0, 0)$?

$z = 2x + 3y$ (is basically the same as the linearization)

(c) Use differentials, or the linearization you found in part (a), to estimate without calculator $f(1.02, 0.99)$.

$f(1.02, 0.99) \approx 2\cdot(1.02) + 3\times(0.99)$

However, this approximation is not very good since the point $(1.02, 0.99)$ is NOT close to $(0,0)$, the reference point of the approximation

There was a mistake on my part in choosing the point for (c).

2. According to the ideal gas law, the pressure, temperature, and volume of a confined gas are related by $P = k\frac{T}{V}$, where k is a constant. Use differentials to approximate the percentage change in pressure if the temperature of a gas is increased 3% and the volume is increased 5%.

We know $\frac{\Delta T}{T} = \frac{dT}{T} = 0.03$ and $\frac{\Delta V}{V} = \frac{dV}{V} = 0.05$.

We want to approximate $\frac{\Delta P}{P}$. But $\frac{\Delta P}{P} \approx \frac{dP}{P}$ (since $\Delta P \approx dP$)

$$dP = \frac{\partial P}{\partial T}dT + \frac{\partial P}{\partial V}dV = \frac{k}{V}dT - \frac{kT}{V^2}dV$$

Thus
$$\frac{dP}{P} = \frac{\frac{k}{V}dT - \frac{kT}{V^2}dV}{k\frac{T}{V}} = \frac{\frac{\cancel{k}}{\cancel{V}}dT}{\cancel{k}\frac{T}{\cancel{V}}} - \frac{\frac{\cancel{kT}}{V^{\cancel{2}}}dV}{\frac{\cancel{kT}}{\cancel{V}}}$$

So
$$\frac{dP}{P} = \frac{dT}{T} - \frac{dV}{V} = 0.03 - 0.05 = \underline{-0.02}$$

Thus, the pressure will decrease by approximately 2%.

3. Use chain rule to find the derivative $df/dt|_{t=0}$, for the function $f(x, y, z) = xy^2z^3$ along the path $< x(t) = e^t \cos t\,,\ y(t) = e^t \sin t\,,\ z(t) = t.>$ at $t = 0$.

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} =$$

$$\frac{df}{dt} = y^2z^3 \cdot (e^t\cos t - e^t \sin t) + 2xyz^3(e^t \sin t + e^t \cos t) + 3xy^2z^2 \cdot 1$$

But at $t=0$ $\quad x(0) = e^0\cos(0) = 1\ ,\ y(0) = e^0 \cdot \sin(0) = 0\ ,\ z(0) = 0$

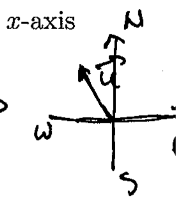
Thus $\displaystyle \frac{df}{dt}\Big|_{t=0} = \boxed{0}$

4. The temperature at a point (x, y) on a metal plate in the xy-plane is given by $T(x, y) = x^2 - \frac{3}{2}y^2 + 20$ degrees Celsius. Assume x, y are measured in centimeters.

(a) Suppose a bug is positioned initially at the point $(1, 1)$ on the plate. What is the temperature at $(1, 1)$?

$$T(1,1) = 1 - \frac{3}{2} + 20 = 19.5\ ^\circ C$$

(b) In which direction should the bug go (from $(1, 1)$) to experience the most rapid *decrease* in temperature? Give your answer as a unit vector, but also as an (approximate) geographical direction. (Assume that the positive x-axis points East and that the positive y-axis points North.)



$(\nabla T)(x,y) = \langle 2x, -3y\rangle$

$(\nabla T)(1,1) = \langle 2, -3\rangle$

Direction of most rapid decrease $= \vec{u} = -\dfrac{\nabla T}{|\nabla T|} = -\dfrac{1}{\sqrt{13}}\langle 2,-3\rangle = \dfrac{1}{\sqrt{13}}\langle -2, 3\rangle$

The bug should go NNW

(c) Suppose next that on the same metal plate there is a heat-seeking ant, which always move in the direction corresponding to greatest increase in temperature. If the ant is initially at ~~the origin~~ (1,1), find the trajectory of this ant.

$\vec{r}(t) = \langle x(t), y(t)\rangle$ position vector of the ant at time t

Want $\vec{r}\,'(t) = k(t)(\nabla T)(x(t), y(t))$ so $\begin{cases} x'(t) = k(t)\cdot 2x(t) \\ y'(t) = k(t)\cdot(-3y(t)) \end{cases}$ with $\begin{matrix} x(0) = 1 \\ y(0) = 1 \end{matrix}$

$$\frac{dy}{dx} \underset{\text{Chain Rule}}{=} \frac{y'(t)}{x'(t)} = \frac{-3y}{2x} \Rightarrow \frac{dy}{-3y} = \frac{dx}{2x} \Rightarrow$$

$$\Rightarrow \int \frac{dy}{-3y} = \int \frac{dx}{2x} \Rightarrow -\frac{1}{3}\ln(y) = \frac{1}{2}\ln|x| + c$$

when $x=1$, $y=1$ so $\quad -\frac{1}{3}\ln(1) = \frac{1}{2}\ln(1) + c \Rightarrow \underline{c = 0}$

Thus $\ln y = -\frac{3}{2}\ln x \Rightarrow \ln y = \ln\left(x^{-\frac{3}{2}}\right) \Rightarrow \boxed{y = x^{-\frac{3}{2}}}$ Trajectory of the heat-seek ant.