MAD 3301 - GRAPH THEORY – Spring 2022 – Review for Final Exam

Final Exam is on Tuesday, April 26, 9:30-11:45am in OE 105. It is a comprehensive exam, so all topics we covered could be tested in some form. Be sure to review your midterm exams, the worksheets and homework assignments.

KEY CONCEPTS AND MAIN DEFINITIONS:

Graph, multi-graph, general graph (graph-like object), digraph, general digraph (digraph-like object), in-degree & out-degree (for digraphs), degree of a vertex, degree sequence, $\delta(G)$, $\Delta(G)$, sub-graphs, regular graphs, adjacency matrix; geometric, set-theoretic, and matrix representation of graphs; isomorphism of graphs; family of graphs (P_n, C_n, K_n,, bipartite graphs and complete bipartite graphs K_{m,n}, circulant graphs); walk, trail, circuit, cycle, path, distances in weighted or unweighted graphs; connected graphs, the connected components of a graph, weakly & strongly connected digraphs, bridge (cut-edge), edge connectivity, vertex connectivity, cut-vertex; trees, forests, non-identical trees, leaves, minimum spanning trees, rooted trees, levels, height of a tree, children, parent, n-ary trees, binary trees, binary coding, uniquely decipherable coding, weighted path-length of a binary coding, optimal binary coding.

Euler circuits, Open Euler trails, Chinese postman problem, minimum postman walk, Hamilton cycles, Hamilton paths, planar graphs, planar embeddings, maximal planar graphs, K_5, $K_{3,3}$, creating & merging out vertices of degree 2, graph homeomorphisms, pieces of a subgraph, segments, spheroidal graphs, stereographical projection, polyhedral graphs, the five regular polyhedra, crossing number of a graph, toroidal graphs, geometric dual of a graph, legal colorings, chromatic number, matchings, maximal matchings, marriages, stable marriages.

MAIN ALGORITHMS:

1. (a) Graphical sequence algorithm
   (b) Graph recovery algorithm,
2. (a) Breadth First Search (BFS) algorithm for finding distances in an unweighted graph.,
   (b) Dijkstra’s distance algorithm in a weighted digraph or a weighted graph,
3. (a) Kruskal's minimum-weight spanning tree algorithm,
   (b) Prim's minimum-weight spanning tree algorithm,
4. (a) Prufer's tree-encoding algorithm,
   (b) Prufer's tree-decoding algorithm,
5. Huffman's optimal-coding algorithm.

6. (a) Fleury's Euler-circuit (& open Euler-trail) algorithm.
   (b) Minimal postman-walk algorithm (Chinese postman algorithm).
7. The basic rules for seeking a Hamiltonian cycle (or showing there is none).
8. Pre-processing graphs for planarity & the DMP planarity algorithm.
9. The basic coloring algorithm (which guarantees a coloring with $\Delta(G) + 1$ colors)
10. The stable marriage algorithm

You should know well the statements of all important theorems and how to use them, even if they are not listed below.

In addition, there will be one or two proof questions selected from the following:

Potential Proof Questions:

1. The number of walks of length n from v_i to v_j = $(A^n)[i, j]$.
2. (a) A connected graph with p vertices has at least p-1 edges.
   (b) A graph with p vertices and more than (p-1)(p-2)/2 edges is always connected.
3. (a) If G is a disconnected graph, then G^c must be connected.
   (b) If G has p vertices and $\delta(G) \geq (p-1)/2$, then G is connected.
4. (a) Any tree with p vertices has exactly p-1 edges.
   (b) G is a tree if and only if there is exactly one path between any two vertices.

5. (a) The connected graph G has an Euler circuit iff each vertex in G is of even degree.
   (b) It has an open Euler trail iff G has exactly two vertices of odd degree. (*Euler's Theorem*)

6. (a) If G is a connected planar graph, then r = q+2-p. (*Euler's formula*)
   (b) If G is a planar graph with k components, then r = q+k+1-p. (*Gen. Euler's formula*)

7. (a) In any planar graph with $p \geq 3$ vertices, we have $q \leq 3p-6$, with equality iff all regions are bounded by three edges.
   (b) Let G be a planar graph with $p \geq 3$ vertices. TFAE: (i) G is maximal planar; (ii) q = 3p-6; (iii) all regions are bounded by three edges.
   (c) Let G be a planar graph with $p \geq 3$ vertices and assume *girth*(G) = $k \geq 3$. Then $q \leq k(p-2)/(k-2)$.

8. The 5 regular polyhedra Theorem.

9. (a) Let G be a graph with at least one edge. Then G is bipartite iff $\chi(G) = 2$ iff G contains no cycle of odd length.
   (b) In any graph G, $\chi(G) \leq \Delta(G) + 1$. (Here $\Delta(G)$ = largest degree in G.)
10. (*Heawood's 5 color theorem*) Any planar graph G admits a legal coloring with 5 colors.