Name: Solution Key PantherID: ____________

Exam 1 – MAD 3301 Graph Theory – Spring 2022

Important Rules:

A. Any electronic device (cell phone, calculator of any kind, smart-watch, etc.) should be turned off at the beginning of the exam and placed in your bag, NOT in your pocket. Electronic items, notes, texts, or formula sheets should NOT be used at any time during the examination. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

B. Unless otherwise mentioned, to receive full credit you must show your work. Answers which are not supported by work might receive no credit.

1. (12 pts) Find, if possible, a graph with degree sequence $< 4, 4, 3, 3, 2, 2 >$ by using the *Graphical Sequence Algorithm.*

$\langle 4, \underline{4}, \underline{3}, \underline{3}, \underline{2}, 2\rangle$ graphical

$\Updownarrow$

$\langle 3, 2, 2, 1, 2\rangle = \langle 3, \underline{2}, \underline{2}, \underline{2}, 1\rangle$ graphical

reorder

$\Updownarrow$

$\langle 1, \underline{1}, 1, 1\rangle$ graphical

$\Updownarrow$

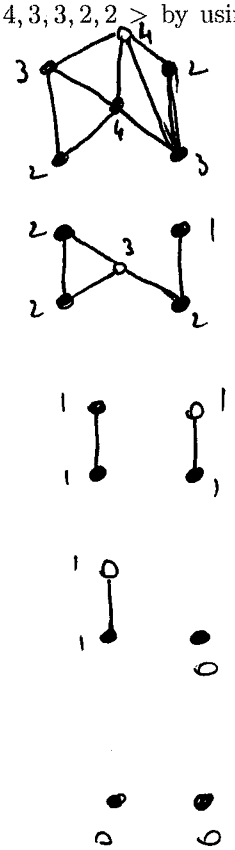
$\langle 0, 1, 1\rangle = \langle 1, 1, 0\rangle$ graphical

reorder

$\Updownarrow$

$\langle 0, 0\rangle$ graphical

True



← realization of $\langle 4, 4, 3, 3, 2, 2\rangle$

4. (12 pts) Give a precise definition to each of the following notions:

(a) A *connected component* H of a connected graph G;

H is a connected component of G if H is a connected subgraph of G which is maximal; that is, if $\tilde{H}$ is a connected subgraph of G so that $H \subseteq \tilde{H} \subseteq G$, then $\tilde{H} = H$.

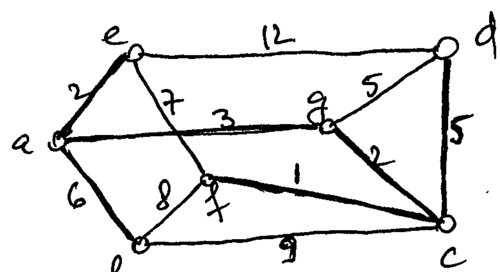
(b) The *height* $h(T)$ of a rooted tree (T, v), where v is the root;

$$h(T) = \max_{x \in V(T)} d(v, x) = \text{eccentricity of the root } v \quad \left(\text{or } h(T) = \text{highest level of the rooted tree}\right)$$

(c) The *vertex connectivity number* $k_v(G)$ of a graph G.

$k_v(G) =$ the minimum number of vertices that can be removed to make G disconnected or K_1.

5. (12 pts) For the graph on the right apply Prim's algorithm starting at vertex d to obtain a minimal weight spanning tree.

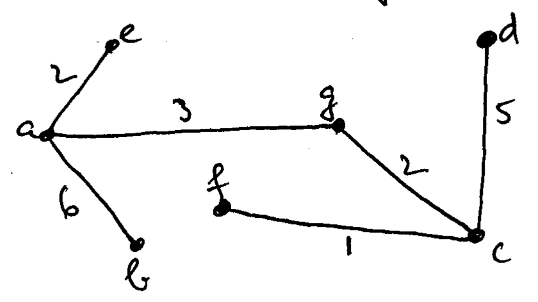


but g could also be chosen

$E(T)$	$V(T)$	$d(\cdot, V(T))$: a	b	c	d	e	f	g	x_0
$\emptyset$	$\{d\}$	∞	∞	5	•	12	∞	5	c
$\{cd\}$	$\{c,d\}$	∞	9	•	•	12	1	2	f
$\{cd, df\}$	$\{c,d,f\}$	∞	8	•	•	7	•	2	g
$\{cd, df, cg\}$	$\{c,d,f,g\}$	3	8	•	•	7	•	•	a
$\{cd, df, cg, ga\}$	$\{a,c,d,f,g\}$	•	6	•	•	2	•	•	e
$\{cd, df, cg, ga, ae\}$	$\{a,c,d,e,f,g\}$	•	6	•	•	•	•	•	b
$\{cd, df, cg, ga, ae, ab\}$	$\{a,b,c,d,e,f,g\} = V(G)$	•	•	•	•	•	•	•	

Minimal weight spanning tree

T



$w(T) = 1 + 2 + 2 + 3 + 5 + 6 = 19$ (not asked)

2. (10 pts) For each of the following circle True or False questions. No justification is necessary. (2 pts each)

(a) For any vertex v in a digraph G, $indeg(v) = outdeg(v)$. **True** **False**

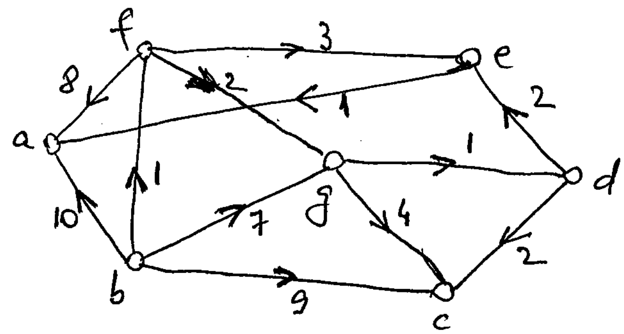
(b) $K_{2,3}$ and $K_{3,2}$ are isomorphic. **True** **False**

(c) $K_{3,3}$ contains a subgraph isomorphic to K_3. **True** **False**

(d) The number of non-identical trees on 10 vertices is 10^8. **True** **False**

(e) If a graph G contains a circuit, then G contains a cycle. **True** **False**

3. (12 pts) Find the distances from vertex b to each of the other vertices of the weighted digraph on the right by using *Dijkstra's Algorithm.*



L(a)	L(b)	L(c)	L(d)	L(e)	L(f)	L(g)	T	i	v_0
∞	0	∞	∞	∞	∞	∞	{a, b, c, d, e, f, g}	0	b
10	•	9	∞	∞	1	7	{a, c, d, e, f, g}	1	f
9	•	9	∞	4	•	3	{a, c, d, e, g}	2	g
9	•	7	4	4	•	•	{a, c, d, e}	3	d (or e could have been chosen
9	•	6	•	4	•	•	{a, c, e}	4	e
5	•	6	•	•	•	•	{a, c}	5	a
•	•	6	•	•	•	•	{c}	6	c
•	•	•	•	•	•	•	∅	STOP	

$d(b,\cdot)=$	5	0	6	4	4	1	3

6. (12 pts) Find the tree that corresponds to the sequence $< 5, 1, 5, 3 >$ via Prüfer's Tree Decoding Algorithm.

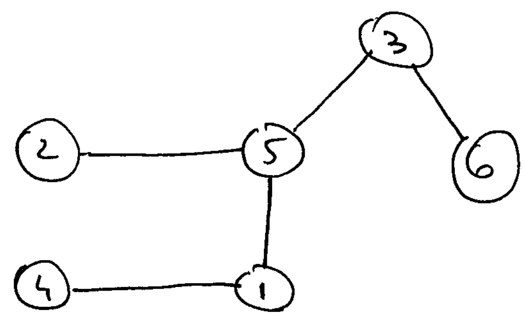
$|\underset{\sim}{s}| = 4 = p-2 \Rightarrow p = 6$ so $V = \{1, 2, 3, 4, 5, 6\}$

$\deg(k) = 1 + \#$ of times k appears in $\underset{\sim}{s}$

Thus $\deg(1) = 2$ $\deg(2) = 1$ $\deg(3) = 2$ $\deg(4) = 1$ $\deg(5) = 3$ $\deg(6) = 1$

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$\ell(i)$	$s(i)$
2	1	2	1	3	1	1	2	5
2	0	2	1	2	1	2	4	1
1	0	2	0	2	1	3	1	5
0	0	2	0	1	1	4	5	3
0	0	1	0	0	1	5	3	6

T



a b c d e

7. (12 pts) The five characters a, b, c, d, e occur with frequencies 25, 10, 15, 15, 35 respectively. Find an optimal binary coding for these five characters and find the weighted-path length of your coding by using Huffman's algorithm. (4 pts bonus) How many different optimal codings are possible in this case?

$\underbrace{10, 15}, 15, 25, 35$ ← two options for the choice of lowest two here (because of the choice of the two 15's)

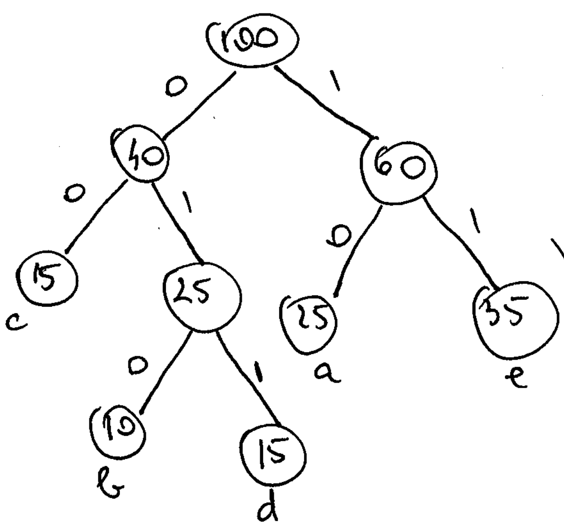
$\underbrace{15, 25}, 25, 35$ ← again two options corresponding on how to choose the 25.

$\underbrace{25, 35}, 40$

$\underbrace{40, 60}$

100

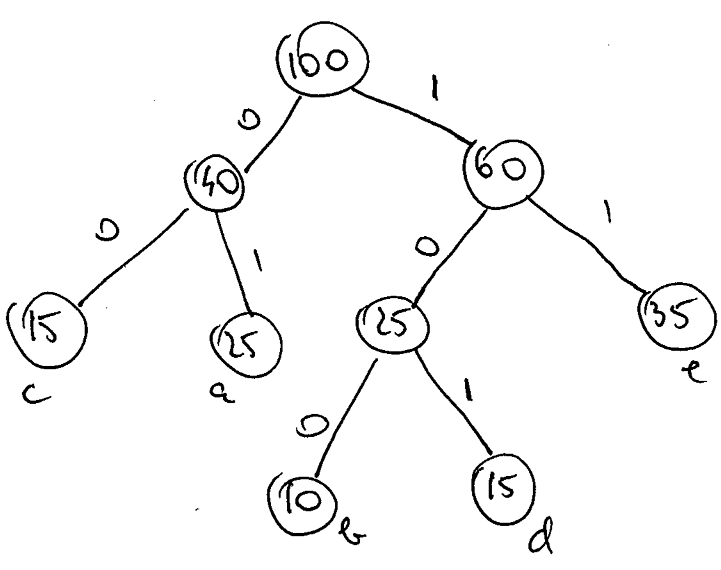
Answer for the bonus first:
There are $2 \times 2 = 4$ possible Huffman's trees in this case, hence 4 different optimal codings



character	a	b	c	d	e
frequency	25	10	15	15	35
code	10	010	00	011	11

$W.P.L = 2\cdot(25) + 3(10) + 2\cdot(15) + 3\cdot(15) + 2\cdot(35) = 225$

with the same tree, but vertices c & d interchanged we get a second possible optimal code



and one more with c and d interchanged.

8. (24 pts) Choose TWO of the following THREE proofs. If you do all three, only top two scores will count while the third score may give you some bonus towards an earlier problem with a lower score.

(A) (12 pts) Let G be a graph and G^c be the complement of G. Prove that if G is a disconnected graph, then G^c must be a connected graph.

(B) (12 pts) Use induction to prove that in any tree with p vertices the number of edges is $p-1$.

(C) (12 pts) Show that if G is a connected graph which contains no vertices of odd degree, then G contains no bridges.

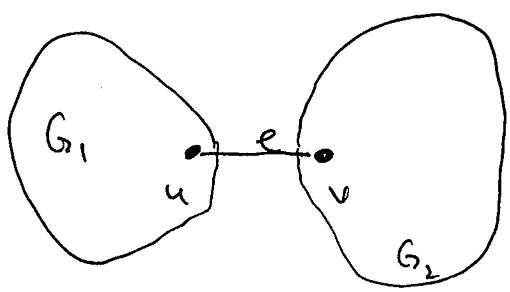
(A) & (B) were done in class – check the notes.

(C) By contradiction: assume that all vertices of G have even degrees and that G contains a bridge e.

Let u, v be the end-points of the bridge edge e

By definition of a bridge $G \setminus e$ is disconnected, so write $G \setminus e = G_1 \cup G_2$, where there is no edge in G (other than e) between a vertex from G_1 and a vertex from G_2.

Assume also that u is a vertex in G_1 and v is a vertex in G_2



In G_1, $\deg_{G_1}(u) = \deg_G(u) - 1$ odd number (by the assumption that all vertices in G have even degree)

if $x \in V(G_1) \setminus \{u\}$

$\deg_{G_1}(x) = \deg_G(x)$ even (as $u \neq x \in V(G_1)$ the only edges from x (in G and G_1) are to vertices in G_1)

Thus, G_1 contains only one vertex of odd degree.

But this contradicts the hand-shake lemma.

Thus, if all vertices in G have even degrees, then G has no bridge.