Name: Solution Key PantherID: ____________

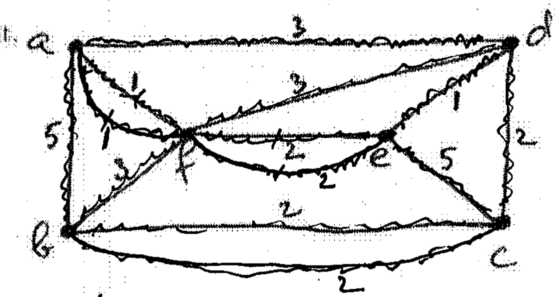
Exam 2 – MAD 3301 Graph Theory – Spring 2022

Important Rules:

A. Any electronic device (cell phone, calculator of any kind, smart-watch, etc.) should be turned off at the beginning of the exam and placed in your bag, NOT in your pocket. Electronic items, notes, texts, or formula sheets should NOT be used at any time during the examination. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

B. Unless otherwise mentioned, to receive full credit you must show your work. Answers which are not supported by work might receive no credit.

1. (15 pts) Find a *minimum postman walk* for the graph on the right, using the *Postman Algorithm*. Also find the *total length* of your minimum postman walk. You can assume that the postman must start and return at vertex a.



Odd degree vertices : {a, b, c, e}

Table of distances

via the path afb

d(x,y)	a	b	c	e
a	0	4	5	3
b		0	2	5
c			0	3
e				0

via the path cde

Odd vertices pairings

{a,b}, {c,e} d(a,b) + d(c,e) = 7

{a,c}, {b,e} d(a,c) + d(b,e) = 10

$\boxed{\{a,e\}, \{b,c\} \quad d(a,e) + d(b,c) = 5}$

optimal pairing.

Add the extra edges af, fe and bc to the original graph and apply Fleury's algorithm to the new (multi)-graph.

A possible postman walk is

afefabcbfdecda

The total length of the minimum postman walk is:

$3\times1 + 5\times2 + 3\times3 + 2\times5 = 32$

2. (15 pts) For each of the following statements circle True or False. No justification is necessary. (3 pts each)

(a) K_n is planar for any $n \geq 3$. **True** **False**

(b) K_n has a Hamiltonian cycle for any $n \geq 3$. **True** **False**

(c) $K_{6,8}$ has an Euler circuit. **True** **False**

(d) Any graph G that is toroidal is also spheroidal. **True** **False**

(e) There is no maximal planar graph G with $q = 20$ edges. **True** **False**

$20 = q = 3p - 6$

$\Rightarrow p = \frac{26}{3}$

3. (15 pts) Prove that in any graph G, $\chi(G) \leq \Delta(G) + 1$.

Proof by induction:

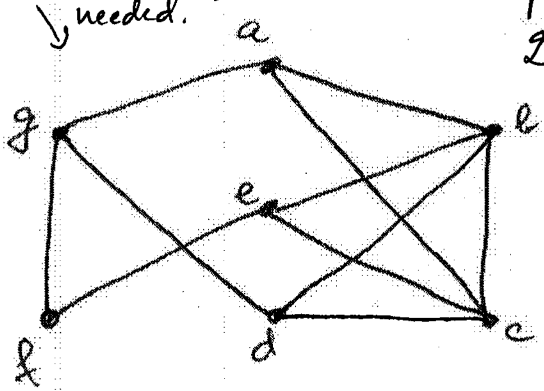
See class notes (first lecture on coloring)

or prof. Ram's notes.

No preliminary preparation needed.

$p = 7$
$q = 11$

4. (15 pts) Determine whether or not the graph G on the right is planar, by using the *DMP Planarity algorithm*. Show the embeddings for each step of the algorithm.



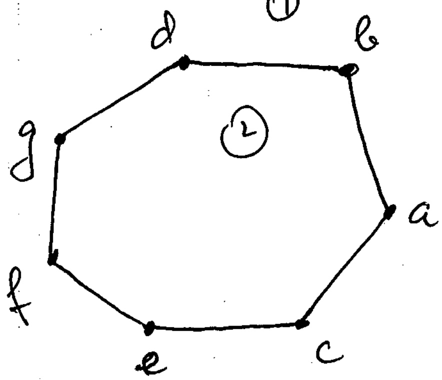
$q = \underline{11} \le 3p - 6$

$3 \cdot 7 - 6 = 15$

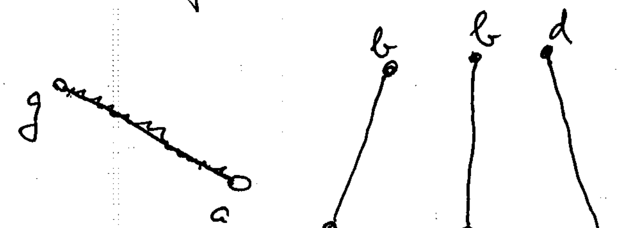
(5 pts bonus) Is the graph G toroidal? Briefly justify.

Your first subgraph H_1 should be a cycle (the longer, the better). This graph actually admits a Hamiltonian cycle, so this will be my choice for $\underline{H_1 : \text{cycle}}$ efgdbace

H_1

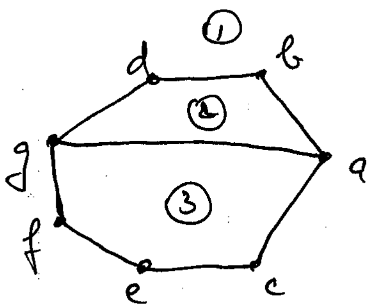


Parts of G relative to H_1

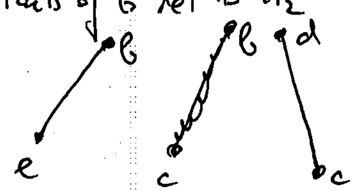


R: {1,2} ← choice   {1,2}   {1,2}   {1,2}.

H_2

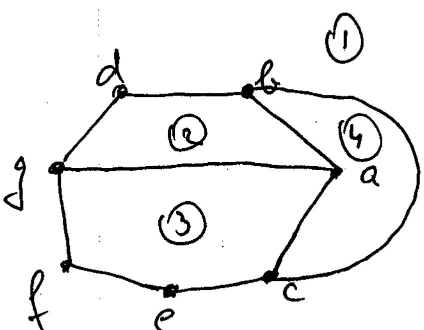


Parts of G rel to H_2

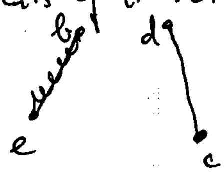


R: {1}   {1} ↑   {1}

H_3



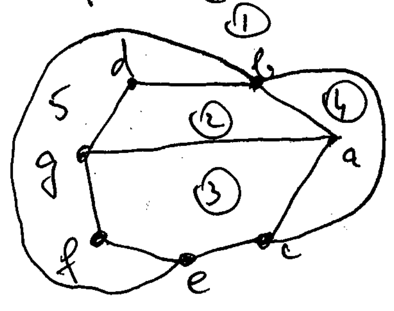
Parts of G rel. to H_3.



R: {1}   {1}

Bonus: G is non-planar but it can be drawn with just one edge crossing. Thus $cr(G) = 1$, so G is toroidal

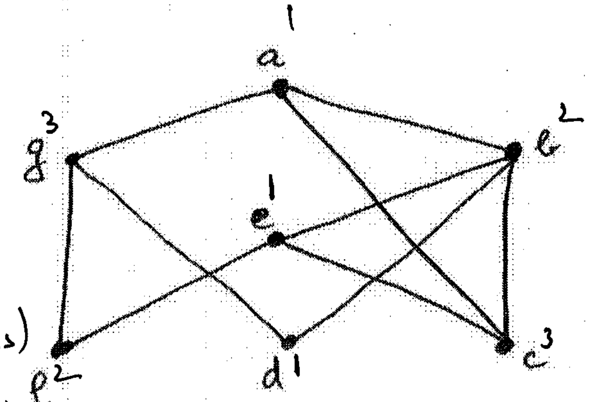
H_4



Part of G rel. to H_4

R = ∅ ← so STOP graph G is NON PLANAR

5. (15 pts) Both parts of this problem refer to the graph G on the right.



(a) (7 pts) Define an (open) Euler trail and determine if the graph G on the right admits an Euler trail. Justify.

A walk covering every edge exactly once (with possible repetition of vertices but not of edges) and starting and ending at different vertices.

No, the graph on the right does NOT have an Euler trail as G contains four vertices of odd degree.

(per Euler's theorem there should be exactly two vertices of odd degree for an open Euler trail to exist)

(b) (8 pts) Find, with justification, $\chi(G)$.

Using the ordering $\{a, b, c, d, e, f, g\}$ and applying the Basic Coloring algorithm, we get a legal coloring of G with 3 colors.

Thus $\chi(G) \leq 3$.

On the other hand G contains triangles (cycles of length 3), thus $\chi(G) \geq 3$.

Hence $\chi(G) = 3$

6. (15 pts) Let G be a planar graph with p vertices and q edges and let $\mathrm{girth}(G) = k \geq 3$.

Show that $q \leq \frac{k(p-2)}{k-2}$. When does equality hold?

This was both a worksheet question and part of your suggested problems for planarity Chapter.

If $\mathrm{girth}(G) = k$, the smallest cycle of G has length k, thus every region in a planar ~~representation~~ embedding of G is bounded by at least k edges

$$2q = \sum_{R \in \text{set of regions}} (\text{\# of edges on boundary of region } R) \geq k \cdot r$$

as each edge is on the boundary of two regions

where $r = \#$ of regions of G.

Thus $r \leq \frac{2q}{k}$

Using this inequality in the Euler formula

$p - q + r = 2$ (which applies as G is planar)

we get $r = 2 - p + q \leq \frac{2q}{k}$

Doing some algebra with the inequality, we get

$$k(2-p) + kq \leq 2q \iff kq - 2q \leq k(p-2)$$

$$\iff (k-2)q \leq k(p-2) \iff q \leq \frac{k(p-2)}{k-2}$$

Equality holds if and only if all regions of G are bounded exactly by k edges.

7. (15 pts) Choose ONE of the following TWO. If you do both, only the top score will count, while the second score may give you some bonus towards an earlier problem with a lower score.

(A) State Ore's Theorem and then use it to prove the following: Let G be a connected graph with $p \geq 3$ vertices. If for any pair of non-adjacent vertices x, y, $\deg(x) + \deg(y) \geq p - 1$, then G has a Hamiltonian path

(B) A soccer ball can be thought of as a convex polyhedron in which all vertices have degree 3, and two regular hexagons and one regular pentagon are incident at each vertex. Find, with mathematical justification, the number of pentagon faces and the number of hexagon faces of a soccer ball.

For (A) see class notes or Prof. Ram's notes

<u>Solution for B.</u>

Start from the Euler formula $p - q + r = 2$ (1)

where $r = r_1 + r_2$, $r_1 = \#$ of pentagon faces

$r_2 = \#$ of hexagon faces.

As all vertices have degree 3, by handshake lemma we get

$3p = 2q$ (2)

Consider next the sum of the vertices on all pentagonal regions.

$$p \underset{\uparrow}{=} \sum_{P \leftarrow \text{pentagonal region}} (\#\text{ of vertices of } P) = 5r_1 \quad . \text{ Thus } p = 5r_1 \quad (3)$$

since every vertex is incident to exactly one pentagon

Similarly, next consider the sum of the vertices on all hexagonal regions

$$2p \underset{\uparrow}{=} \sum_{\mathcal{H} \leftarrow \text{hexagonal region}} (\#\text{ of vertices of } \mathcal{H}) = 6r_2 \quad . \text{ Thus } p = 3r_2 \quad (4)$$

since each vertex is incident to <u>two</u> hexagons.

Using (2), (3), (4) back in (1), one gets (after some algebra) $p = 60$, $q = 90$

$r_1 = 12 \leftarrow \#$ of pentagons

$r_2 = 20 \leftarrow \#$ of hexagons

Check $r = r_1 + r_2 = 32$ and $p - q + r = 60 - 90 + 32 = \boxed{2}$