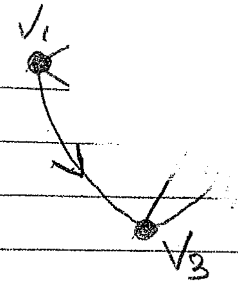
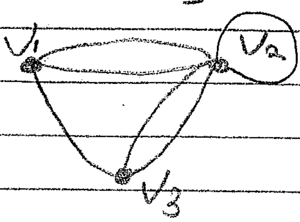
10/10

1- G



a) $\tilde{G}$



b) $A_G = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad A_{\tilde{G}} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

- Show that $\underline{A_{\tilde{G}} = A_G + (A_G)^T}$

$$(A_G)^T = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$A_{\tilde{G}} = A_G + (A_G)^T$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0+0 & 2+0 & 1+0 \\ 0+2 & 1+1 & 1+1 \\ 0+1 & 1+1 & 0+0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix}}$$

Does this hold for any general digraph G with supporting grap $\tilde{G}$?

• Let G be a general digraph and let G' be a graph constructed from G by inverting the direction of every one of its edges.

Then, if A_G is the adjacency matrix of G, $A_{G'} = (A_G)^T$ is the adjaceency matrix of G'.

Now, if we compute $A_G + (A_G)^T$ we would get an adjacency matrix corresponding to a graph with all its edges directed in both directions, which we can assume means undirected edges.

Therefore, that graph would be $\tilde{G}$ and Good!

$$\boxed{A_{\tilde{G}} = A_G + (A_G)^T, \ \forall \text{ directed graph } G.}$$

c) The number of directed walks of length 3 from v_1 to v_3 is given by $(A_G)^3[1,3]$.

$$A_G = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (A_G)^2 = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(A_G)^2 = \begin{pmatrix} 0\cdot0+2\cdot0+1\cdot0 & 0\cdot2+2\cdot1+1\cdot1 & 0\cdot1+2\cdot1+1\cdot0 \\ 0\cdot0+1\cdot0+1\cdot0 & 0\cdot2+1\cdot1+1\cdot1 & 0\cdot1+1\cdot1+1\cdot0 \\ 0\cdot0+1\cdot0+0\cdot0 & 0\cdot2+1\cdot1+0\cdot1 & 0\cdot1+1\cdot1+0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$(A_G)^3 = (A_G)^2 \circ A_G$$

$$= \begin{pmatrix} 0 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot0+3\cdot0+2\cdot0 & 0\cdot2+3\cdot1+2\cdot1 & 0\cdot1+3\cdot1+2\cdot0 \\ 0\cdot0+2\cdot0+1\cdot0 & 0\cdot2+2\cdot1+1\cdot1 & 0\cdot1+2\cdot1+1\cdot0 \\ 0\cdot0+1\cdot0+1\cdot0 & 0\cdot2+1\cdot1+1\cdot1 & 0\cdot1+1\cdot1+1\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 5 & 3 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$(A_G)^3[1,3] = 3$

there are 3 walks of length 3 from v_1 to v_3.

d)

$d_G(x,y)$	v_1	v_2	v_3
v_1	0	1	1
v_2	$+\infty$	0	1
v_3	$+\infty$	1	0

✓

$d_{\tilde{G}}(x,y)$	v_1	v_2	v_3
v_1	0	1	1
v_2	1	0	1
v_3	1	1	0

✓

2- (i) $d(x,y) \geq 0$, with $d(x,y) = 0$ iff $x = y$.

From the definition of the distane function, it's image is $[0, +\infty]$, so $d(x,y) \geq 0 \ \forall x,y$.

10|10

$d(x,y)$ is the length of the shortest path between x and y.

- $d(x,y) = 0 \implies x = y$

this is true because if there exists a path of length 0, it most be the null path, thus we are computing the distance from x to x or y to y, in any case $x = y$.

- $x = y \implies d(x,y) = 0$

$x = y$ implies we are computing the distanee from x to x or y to y and there can be many paths between them, but definitely the shortest one is the null path of length 0.

(ii) $d(x,y) = d(y,x)$

Since the edges are undirected a path from x to y woold have the same minimum length as a path from y to x.

(iii) $d(x,z) \leq d(x,y) + d(y,z)$

$d(x,y)$ and $d(y,z)$ are the lengths of the shortest paths between x and y and y and z respectively.

So $d(x,y) + d(y,z)$ is the length of a walk between x and z because there might be repeating vertices or edges in the middle or it might just not be the shortest distance from x to z.

Now, a path can always be extracted from a walk, which will be the shortest path from x to z.

Thus:

a) if said walk is the shortest path between x and z then $d(x,z) = d(x,y) + d(y,z)$

b) if not, then the shortest path from x to z is shorter than that walk: $d(x,z) < d(x,y) + d(y,z)$

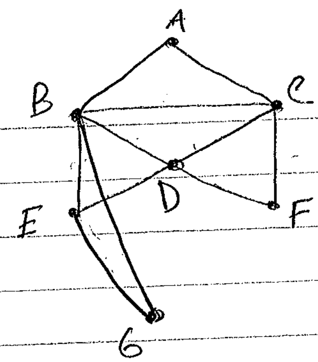
As either a) or b) could occur

~~Assuming both a) and b)~~:

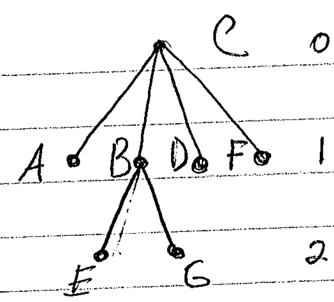
$$\boxed{d(x,z) \leq d(x,y) + d(y,z)}$$

3-

10/10



a)



b) Different search trees are correct starting from the same vertex because the amount of edges would be the same and so will the distances from the starting vertex to all the other ones. It all depends on which vertices are put first on the search tree. -

4- Modified BFS algorithm to find the length of the smallest cycle starting at any given vertex x of graph G

<u>Algorithm:</u>

<u>Input:</u> An unlabeled graph $G=(V,E)$ with distinguished vertex x.

<u>Output:</u> The length of the smallest cycle starting from x in G.

<u>Method:</u> Use a variable i to measure the distance from x, and label vertices with i as their distance is found:

Steps: Excellent! You likely found the easiest solution!

1) $i=0$

2) Label x with i Great idea!

9+4

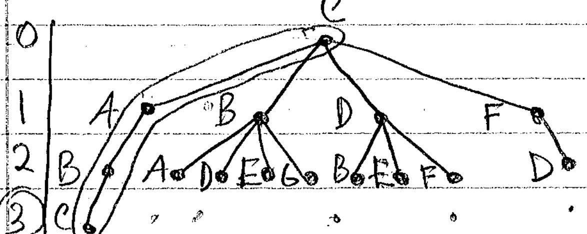
3) Find all vertices (labeled and unlabeled) adjacent to at least one vertex with label i (except the vertex it comes from).
4) Label all vertices found in step 3 with $i+1$.

5) If vertex x has appeared again in the produced search tree, stop.
the length of the smallest cycle will be the label of vertex x the first time it reappeared.
If $i = p+1$ then STOP and say NO cycle. Otherwise,
6) Let $i=i+1$ ~~and~~ go to step 3).

* In order to list that cycle take the path from the starting X vertex to the first reappearance of vertex X and list it as $(X, \ldots, X)$.
the cycle is now listed.

OK, but how exactly would the computer know to list that cycle?

Example: From the graph in 3. Find the length and list the smallest cycle starting from vertex C.



• the smallest cycle has length 3 and it is (C, A, B, C)

* this cycle is not unique, if we continue we will find another 5 cycles of length 3 which could have been the smallest one.