Midterm Exam - MAS 5145 Fall 2011 **NAME:** ____________________

To receive credit you MUST SHOW ALL YOUR WORK.

1. (16 pts - 4 pts each part):

(a) Suppose $L : V \to W$ is a linear map between vector spaces. Define $Range(L)$ and $rank(L)$.

(b) If V is a vector space over a field $\mathbb{K}$, define the dual space V^*.

(c) State the Fundamental Isomorphism Theorem.

(d) Suppose that U and W are subspaces of a vector space V. What does it mean that $V = U \oplus W$?

2. (14 pts) Consider a vector space V and a collection of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ in V. Show that $\text{Span}\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a subspace of V.

3. (16 pts) (a) (4 pts) If A is an $n \times n$ matrix, define the trace of A, $Tr(A)$, and state the relationship between $Tr(A)$ and the eigenvalues of A (no proof required).

(b) (6 pts) Prove that $Tr(AB) = Tr(BA)$ for any two $n \times n$ matrices A, B.

(c) (6 pts) Show that if A, B are conjugate (similar) matrices, then $Tr(A) = Tr(B)$.

4. (14 pts) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Is the matrix A diagonalizable? Justify. Compute e^A.

5. (20 pts) Diagonalize the operator $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ defined by $L(\mathbf{p})(t) = \mathbf{p}(a - t)$, where a is a given constant.

Bonus 6 pts: Is there a basis in $\mathbb{R}[t]$ that diagonalizes the operator $L : \mathbb{R}[t] \to \mathbb{R}[t]$, $L(\mathbf{p})(t) = \mathbf{p}(a - t)$?

6. (20+10 pts) Let L_1, L_2 be linear operators on a finite dimensional vector space V. Assume that $L_1 \circ L_2 = L_2 \circ L_1$. Prove each of the following:

(a) (6 pts) $Ker(L_1) + Ker(L_2) \subseteq Ker(L_1 \circ L_2)$.

(b) (7 pts) If L_1 and L_2 are diagonalizable, then $Ker(L_1) + Ker(L_2) = Ker(L_1 \circ L_2)$.

(c) (7 pts) If L_1 and L_2 are not diagonalizable, then it is possible that $Ker(L_1) + Ker(L_2) \neq Ker(L_1 \circ L_2)$.
Hint: What is A^2 for the matrix A in Pb. 4?

(d)* (10 pts) (This part is for take home, as it is more difficult. However, if you can do it in class, you will receive even more bonus points.) Investigate whether $Ker(L_1) + Ker(L_2) = Ker(L_1 \circ L_2)$ assuming that only one of the operators is diagonalizable.