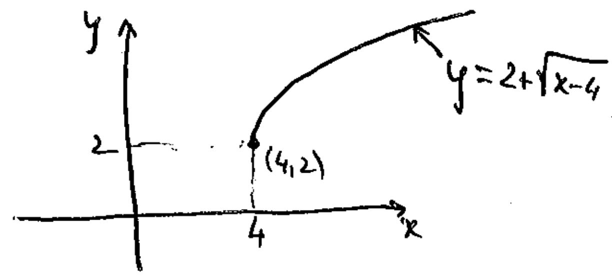
Exam 1- MAC 2311, F'13 **NAME:** Solution Key **Panther ID:** ______

General Directions: *Read the problems carefully and provide answers exactly to what is requested. Use complete sentences and use notation correctly. Incomprehensible work is worthless. I am grading the work, not just the answer. Don't rush, don't try to do too many steps of a computation at once; work carefully. Good luck!*

1. (12 pts) Consider the function $f(x) = 2 + \sqrt{x-4}$.

(a) (4 pts) Sketch a graph of this function.



(b) (4 pts) What is the domain of f? What is the range of f? Write your answers using interval notation.

Domain: $x \in [4, +\infty)$ $\quad x \geq 4$

Range: $y \in [2, +\infty)$

(c) (4 pts) Find a formula for the inverse function $f^{-1}(x)$.

$y = 2 + \sqrt{x-4}$; we should solve for x in terms of y

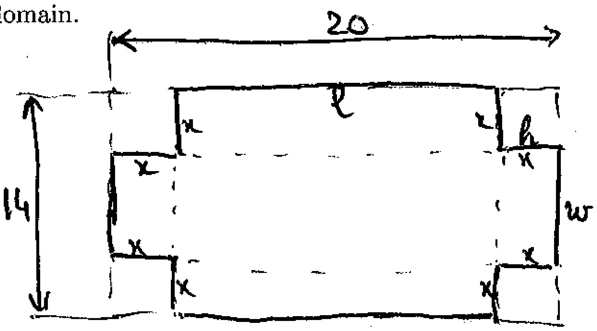
$y - 2 = \sqrt{x-4}$ square both sides

$(y-2)^2 = x - 4 \Rightarrow x = 4 + (y-2)^2$

Now swap x with y (so that the input for the inverse is still x)

$y = 4 + (x-2)^2$. So $f^{-1}(x) = 4 + (x-2)^2$, Note that Domain f^{-1} = Range f = $[2, +\infty)$

2. (8 pts) An open box is to be made from a 14-inch by 20-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (picture will be drawn on board). Let $V(x)$ be the volume of the box that results when the squares have sides of length x. Find the formula for the function $V(x)$ and determine its domain.



$V = l \cdot w \cdot h$

$h = x$

$l = 20 - 2x$

$w = 14 - 2x$

Thus $V(x) = (20-2x)(14-2x)x$

Since all the sides of the box should have non-negative lengths, the domain of $V(x)$ is $x \in [0, 7]$ (so that $14 - 2x \geq 0$)

3. (28 pts) Find the following limits. If the limit is infinite or does not exist, specify so.

(a)(5pts) $\lim_{x\to 3} \frac{x^2-9}{|x-3|} = \lim_{x\to 3} \frac{(x-3)(x+3)}{|x-3|}$ $\frac{0}{0}$ case

$$\lim_{x\to 3^+} \frac{(x-3)(x+3)}{|x-3|} = \lim_{x\to 3^+} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = 6$$

$$\lim_{x\to 3^-} \frac{(x-3)(x+3)}{|x-3|} = \lim_{x\to 3^-} \frac{\cancel{(x-3)}(x+3)}{-\cancel{(x-3)}} = -6$$

Thus, the 2-sided limit D.N.E.

(b)(5pts) $\lim_{x\to 2^-} \frac{x-1}{x-2} = \frac{1}{0^-} = \boxed{-\infty}$

(c)(5pts) $\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(2x)} =$ $\frac{0}{0}$ case

$$= \lim_{x\to 0} \frac{x \cdot \frac{\sin(5x)}{5x} \cdot 5x}{\left(\frac{\tan(2x)}{2x}\right)^2 \cdot (2x)^2} =$$

$$= \lim_{x\to 0} \left[\frac{\frac{\sin(5x)}{5x}}{\left(\frac{\tan(2x)}{2x}\right)^2} \cdot \frac{5\cancel{x^2}}{4\cancel{x^2}} \right] = \boxed{\frac{5}{4}}$$

(d)(5pts) $\lim_{x\to +\infty} \frac{1+\cos(2x)}{x} = 0$ Justification:

As $x \to +\infty$, denominator gets large, numerator oscillates, so think of Squeeze Theorem

$$-1 \le \cos(2x) \le 1 \Rightarrow$$

$$0 \le 1+\cos(2x) \le 2 \xRightarrow{x>0}$$

$$0 \le 1+\cos(2x) \le \frac{2}{x}$$

$x \to +\infty$ $x \to +\infty$

0

(e)(8pts) $\lim_{x\to +\infty} (\sqrt{x^2+100x} - x) = *$

exceptional case $+\infty - \infty$

Idea: Multiply up & down by conjugate

$$= \lim_{x\to +\infty} \frac{(\sqrt{x^2+100x} - x)(\sqrt{x^2+100x} + x)}{\sqrt{x^2+100x} + x} =$$

$$= \lim_{x\to +\infty} \frac{\cancel{x^2} + 100x - \cancel{x^2}}{\sqrt{x^2+100x} + x} = \lim_{x\to +\infty} \frac{100x}{\sqrt{x^2\left(1+\frac{100}{x}\right)} + x} \underset{x>0}{=} \lim_{x\to +\infty} \frac{100x}{x\sqrt{1+\frac{100}{x}} + x}$$

$$= \lim_{x\to +\infty} \frac{100\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{100}{x}} + 1\right)} = \frac{100}{1+1} = \frac{100}{2} = \boxed{50}$$

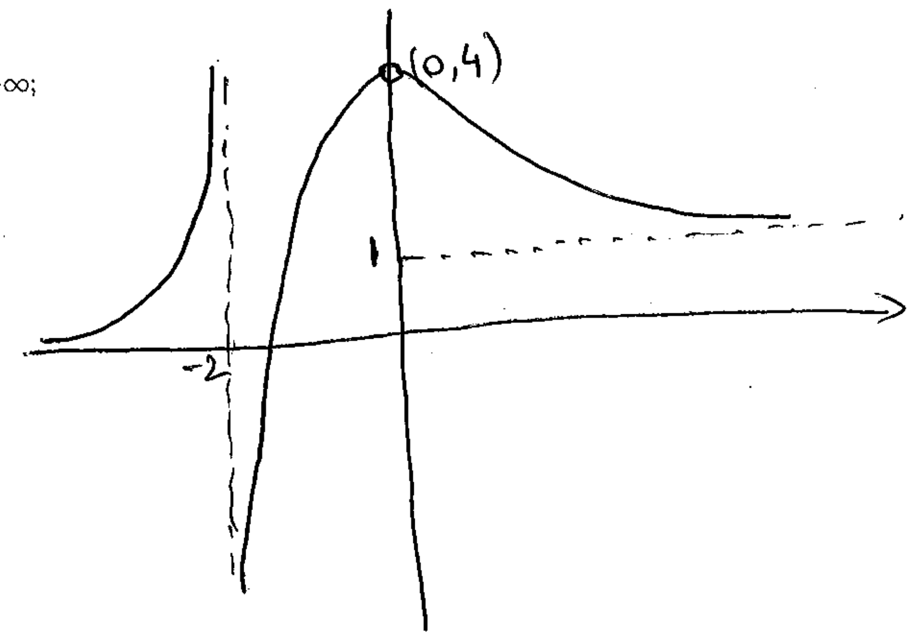
4. (12 pts) Sketch the graph of a function $f(x)$ satisfying all of the following conditions.

(i) $f(x)$ is continuous everywhere except $x = -2$ and $x = 0$;
f is not defined $x = -2$ and $x = 0$;

(ii) $\lim_{x\to -2^-} f(x) = +\infty, \quad \lim_{x\to -2^+} f(x) = -\infty;$

(iii) $\lim_{x\to 0} f(x) = 4;$

(iv) $\lim_{x\to -\infty} f(x) = 0, \quad \lim_{x\to +\infty} f(x) = 1.$



5. (8 pts) Given the function below

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

(a) (4 pts) Is $f(x)$ continuous everywhere? Justify your answer.

The only possible problem point might be $x = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2-1) = 0$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 1$

Note that $f(1) = 1^2 - 1 = 0$

$\Rightarrow$ so $\lim_{x\to 1} f(x)$ D.N.E,

so $f(x)$ is not continuous at $x=1$

(b) (4 pts) Is $f(x)$ differentiable at $x = 1$? Justify your answer.

No, since $f(x)$ is not continuous at $x=1$ it cannot be differentiable at $x=1$.

(If it were differentiable at $x=1$, by the theorem we proved in class, it would have been continuous at $x=1$.)

6. (12 pts) A stone is dropped (with zero initial velocity) from the top of a building. The height above the ground $h(t)$ of the stone after t seconds since it was dropped is given by $h(t) = 128 - 16t^2$ feet.

(a) (2 pts) How tall is the building?

$$h(0) = 128 \text{ ft}$$

(b) (2 pts) When does the stone hit the ground?

$t = ?$ $h(t) = 0$

$0 = 128 - 16t^2 \Rightarrow 16t^2 = 128 \Rightarrow t^2 = \frac{128}{16} = 8 \Rightarrow t = \sqrt{8} = 2\sqrt{2}\,\text{s}.$

(c) (4 pts) Find the average velocity of the stone during the first two seconds of its fall. Give units for your answer.

$V_{avg.} = \frac{h(2) - h(0)}{2 - 0} = \frac{(128 - 16 \cdot 2^2) - 128}{2} = \frac{-64}{2} = -32 \frac{\text{ft}}{\text{s}}$ "$-$" since the stone is dropping

(d) (4 pts) Find the instantaneous velocity at 2 seconds. Give units for your answer.

$V_{inst.\,at\,t=2} = h'(2) = \lim_{\varepsilon \to 0} \frac{h(2+\varepsilon) - h(2)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(128 - 16(2+\varepsilon)^2) - (128 - 16 \cdot 2^2)}{\varepsilon}$

$= \lim_{\varepsilon \to 0} \frac{(128 - 64 - 64\varepsilon - \varepsilon^2) - (128 - 64)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{-64\varepsilon - \varepsilon^2}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{-\cancel{\varepsilon}(64 + \varepsilon)}{\cancel{\varepsilon}_1} = \boxed{-64 \frac{\text{ft}}{\text{s}}}$

Also acceptable $h'(t) = (128 - 16t^2)' = -32t$

so $\boxed{h'(2) = -32 \cdot 2 = -64 \frac{\text{ft}}{\text{sec}}}$

7. (10 pts) (a) (4 pts) Write the general (ϵ, δ) definition for $\lim_{x \to a} f(x) = L$.

For any $\varepsilon > 0$, we can find $\delta > 0$ so that if $|x - a| < \delta$, but $x \neq a$, we have $|f(x) - L| < \varepsilon$.

(b) (6 pts) Use the (ϵ, δ) definition to prove $\lim_{x \to 2} (500x - 3) = 997$.

$|f(x) - L| = |500x - 3 - 997| = |500x - 1000| = 500|x - 2|$

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{500}$

If $|x - 2| < \delta = \frac{\varepsilon}{500} \Rightarrow |f(x) - L| = 500|x - 2| < 500\delta = \cancel{500} \frac{\varepsilon}{\cancel{500}} = \varepsilon$

8. (10 pts) Let $f(x) = \frac{1}{x}$. Find $f'(x)$ using the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{\cancel{x} - \cancel{x} - h}{x(x+h)}}{\frac{h}{1}} =$$

$$= \lim_{h\to 0} \left(\frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} \right) = \boxed{-\frac{1}{x^2}}$$

9. (10 pts) Show that the equation $2\cos x = \tan x$ has a real solution in the interval $(0, \pi/2)$.
Hint: You can do this either using IVT, or, in this case, you may even find the explicit solution (it's OK to leave it as $x = \arcsin(...)$, but justify that waht you get is indeed in the first quadrant).

Sol.1 - without IVT

$$2\cos x = \tan x \iff 2\cos x = \frac{\sin x}{\cos x} \Rightarrow$$

$$\Rightarrow 2\cos^2 x = \sin x \Rightarrow 2(1-\sin^2 x) = \sin x$$

$$\Rightarrow 2 - 2\sin^2 x = \sin x \Rightarrow$$

$$2\sin^2 x + \sin x - 2 = 0$$

Let $w = \sin x$

$$2w^2 + w - 2 = 0$$

$$w_{1,2} = \frac{-1 \pm \sqrt{1-4\cdot 2(-2)}}{4} = \frac{-1 \pm \sqrt{17}}{4}$$

Thus $\sin x = \frac{-1+\sqrt{17}}{4} \Rightarrow \boxed{x = \arcsin\left(\frac{-1+\sqrt{17}}{4}\right) \in \left(0, \frac{\pi}{2}\right) \text{ since } \frac{-1+\sqrt{17}}{4} > 0}$

$\sin x = \frac{-1-\sqrt{17}}{4} \Rightarrow x = \arcsin\left(\frac{-1-\sqrt{17}}{4}\right) \in \left(-\frac{\pi}{2}, 0\right)$

Sol. 2 - using IVT

Let $f(x) = 2\cos x - \tan x$
continuous on $\left[0, \frac{\pi}{2}\right)$

$$f(0) = 2\cos 0 - \tan 0 = 1 - 0 = 1 > 0$$

$$f\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} - \tan\frac{\pi}{6} = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} - \frac{1}{\sqrt{3}} > 0$$

$$f\left(\frac{\pi}{4}\right) = 2\cos\frac{\pi}{4} - \tan\frac{\pi}{4} = \cancel{2}\frac{\sqrt{2}}{\cancel{2}} - 1 > 0$$

$$f\left(\frac{\pi}{3}\right) = 2\cos\frac{\pi}{3} - \tan\frac{\pi}{3} = \cancel{2}\cdot\frac{1}{\cancel{2}} - \sqrt{3} < 0$$

Since $f\left(\frac{\pi}{4}\right) > 0$, $f\left(\frac{\pi}{3}\right) < 0$ and f is continuous on $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, by IVT there exists a value $x_1 \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ so that $f(x_1) = 0$. Done.

Combining the two solutions, we learned that $\left(\frac{-1+\sqrt{17}}{4}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$