Name: Solution Key **Panther ID:** ____________

Exam 1 - MAC2311 - SummerB 2017

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (14 pts) These are True or False questions. No justification required. No partial credit. 2 points each.

(i) If $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ then $f(x)$ is continuous at $x=3$. **True** **False** — $f(3)$ may not equal $\lim_{x\to 3} f(x)$

(ii) $\lim_{x\to-\infty}(x^3+100x^2) = -\infty$ **True** **False** — Junk Rule

(iii) For all $x \neq 0$, $\frac{\tan x}{x} = 1$ **True** **False** (there is no $\lim_{x\to 0}$)

(iv) The function $f(x) = \cot x$ is defined and is continuous for all real numbers x. **True** **False** — $\cot x = \frac{\cos x}{\sin x}$ not defined at $k\pi$, $k \in \mathbb{Z}$

(v) If $\lim_{x\to a} f(x) = 5$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}\frac{f(x)}{g(x)}$ D.N.E. **True** **False** — $\frac{5}{\infty} = 0$

(vi) $0\cdot\infty$ is a limit indeterminate form (or exceptional case). **True** **False**

(vii) If $f(x)$ is a continuous function for all real numbers x, then f has no vertical asymptotes. **True** **False**

2. (12 pts) A robot moves in the positive direction along a straight line so that after t ~~minutes~~ seconds its distance is $s = 2t^3$ feet from the origin.

(a) (5 pts) Find the average velocity of the robot in the time interval $1 \le t \le 3$ seconds. Give units to your answer.

$$V_{ave} = \frac{s(3)-s(1)}{3-1} = \frac{2\cdot 3^3 - 2\cdot 1^3}{2} = \frac{54-2}{2} = 26\ \frac{ft}{s}$$

(b) (7 pts) Use limits to find the instantaneous velocity of the robot when $t=3$ seconds. Give units to your answer.

$$V(3) = \lim_{h\to 0}\frac{s(3+h)-s(3)}{h} = \lim_{h\to 0}\frac{2\cdot(3+h)^3 - 2\cdot 3^3}{h} = \lim_{h\to 0}\frac{2\left[(3+h)^3 - 3^3\right]}{h}$$

$$= \lim_{h\to 0}\frac{2\cdot[\cancel{3}+\cancel{h}-\cancel{3}]\cdot[(3+h)^2+(3+h)\cdot 3+3^2]}{\cancel{h}_{1}}$$ (used the identity $A^3-B^3 = (A-B)(A^2+AB+B^2)$)

$$= 2\cdot[3^2+3^2+3^2] = 2\cdot 27 = \boxed{54\ \frac{ft}{s}}$$

3. (8 pts) Given the function $g(x) = \begin{cases} kx^2+1 & \text{if } x<2 \\ 2 & \text{if } x=2 \\ 2x+k & \text{if } x>2 \end{cases}$

find, if possible, a value of the constant k which will make $g(x)$ continuous everywhere. Justify your answer.

The only problem point is $x=2$.

We want $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2)$

$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (kx^2+1) = k\cdot 2^2+1 = 4k+1$

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (2x+k) = 2\cdot 2+k = k+4$

$g(2) = 2$

So, we want $4k+1 = k+4 = 2$ which is not possible.

Thus, there is no value of k which makes $g(x)$ continuous at $x=2$.

4. (30 pts) Find the following limits. If the limit is infinite or does not exist, specify so (5pts each).

(a) $\lim_{x\to -2} \frac{x^2-2x-8}{8-2x^2} = \frac{0}{0}$

$= \lim_{x\to -2} \frac{(x+2)(x-4)}{2(4-x^2)} =$

$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x-4)}{2(2-x)\cancel{(2+x)}}$

$= \frac{-2-4}{2(2-(-2))} = -\frac{6}{8} = \boxed{-\frac{3}{4}}$

(b) $\lim_{x\to +\infty} \frac{x^2-2x-8}{8-2x^2} \underset{\text{Junk Rule}}{=} \lim_{x\to +\infty} \frac{x^2}{-2x^2} = \boxed{-\frac{1}{2}}$

(c) $\lim_{x\to 3} \frac{|x-3|}{x^2-6x+9} = \frac{0}{0}$

$= \lim_{x\to 3} \frac{|x-3|}{(x-3)^2} = \lim_{x\to 3} \frac{\cancel{|x-3|}^{\,1}}{|x-3|^{\cancel{2}}}$

$= \lim_{x\to 3} \frac{1}{|x-3|} = \frac{1}{0^+} = \boxed{+\infty}$

(d) $\lim_{x\to -2^-} \frac{x-1}{x+2} = \frac{-3}{0^-} = \boxed{+\infty}$

(e) $\lim_{x\to 0} \frac{\tan(7x)}{\sin(2x)+\sin(3x)} \overset{\frac{0}{0}}{=}$

$$= \lim_{x\to 0} \frac{\frac{\tan(7x)}{7x}\cdot 7x}{\frac{\sin(2x)}{2x}\cdot 2x + \frac{\sin(3x)}{3x}\cdot 3x}$$

$$= \lim_{x\to 0} \frac{\frac{\tan(7x)}{7x}\cdot 7\cancel{x}}{\cancel{x}\left[\frac{\sin(2x)}{2x}\cdot 2 + \frac{\sin(3x)}{3x}\cdot 3\right]}$$

$$= \frac{7}{2+3} = \boxed{\frac{7}{5}}$$

(f) $\lim_{x\to+\infty} (\sqrt{x^2+5x}-x) \overset{\infty-\infty}{=}$

$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2+5x}-x)(\sqrt{x^2+5x}+x)}{\sqrt{x^2+5x}+x} =$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2}+5x-\cancel{x^2}}{\sqrt{x^2+5x}+x} = \lim_{x\to+\infty} \frac{5x}{|x|\sqrt{1+\frac{5}{x}}+x}$$

$$\underset{x>0}{=} \lim_{x\to+\infty} \frac{5\cancel{x}}{\cancel{x}\left[\sqrt{1+\frac{5}{x}}+1\right]} = \boxed{\frac{5}{2}}$$

5. (10 pts) Use the Intermediate Value Theorem to show that the equation $x^4+3x-1=0$ has at least two real solutions and locate each solution in an interval of length 0.5. Justify carefully.

Let $f(x)=x^4+3x-1$. It's a polynomial so it is continuous everywhere.

$f(0)=-1<0$
$f(\frac{1}{2})=(\frac{1}{2})^4+3\cdot\frac{1}{2}-1>0$
$\rightarrow$ by IVT there is a value $x_1\in[0,\frac{1}{2}]$ so that $f(x_1)=0$.

$f(-1)=(-1)^4+3\cdot(-1)-1=-3<0$
$f(-2)=(-2)^4+3\cdot(-2)-1=9>0$
by IVT, there is a second solution in the interval $[-1,-2]$

To locate the second solution in an interval of length 0.5 we use the bisection technique. (Good to use $-\frac{3}{2}$ instead of -1.5)

$$f(-\tfrac{3}{2})=(-\tfrac{3}{2})^4+3\cdot(-\tfrac{3}{2})-1=\frac{81}{16}-\frac{9}{2}-1=-\frac{7}{16}<0$$

$f(-2)=9>0$

So by IVT, there is $x_2\in[-2,-\frac{3}{2}]$ so that $f(x_2)=0$.

Thus the equation has at least two real solutions, one in the interval $[0,\frac{1}{2}]$ the other in the interval $[-2,-\frac{3}{2}]$.

6. (12 pts) Sketch the graph of ONE function $f(x)$ satisfying ALL of the following conditions.

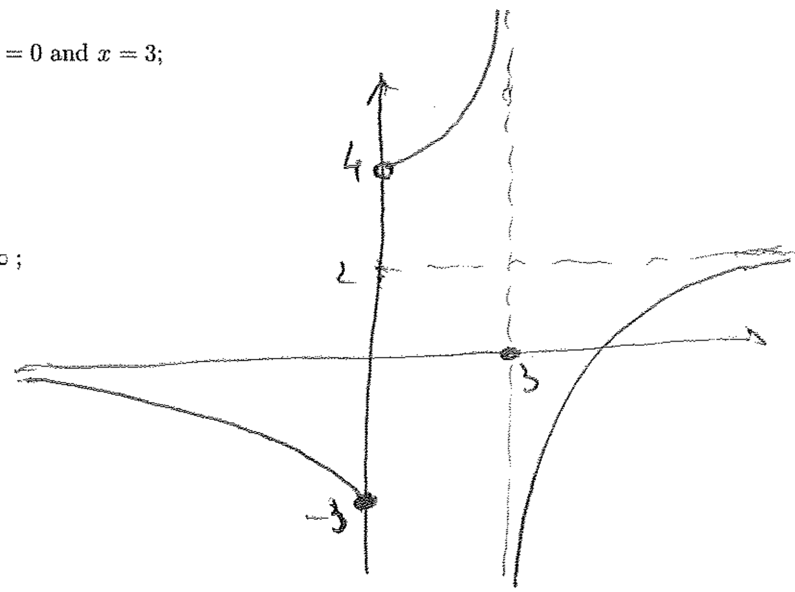
(i) The function is defined for all real numbers;

(ii) The function is continuous everywhere except $x = 0$ and $x = 3$;

(iii) $\lim_{x \to 0^-} f(x) = -3$, $f(0) = -3$, $\lim_{x \to 0^+} f(x) = 4$;

(iv) $\lim_{x \to 3^-} f(x) = +\infty$, $f(3) = 0$, $\lim_{x \to 3^+} f(x) = -\infty$;

(v) $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = 2$.



7. (a) (3 pts) Write the general (ϵ, δ) definition for $\lim_{x \to a} f(x) = L$.

For any $\varepsilon > 0$ there exists $\delta > 0$ so that
if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Choose ONE of the parts (b) and (c). Only ONE will receive credit. Note the different point values.

(b) (7 pts) Use the (ϵ, δ) definition to prove $\lim_{x \to 3} (5x - 7) = 8$.

(c) (12 pts) Use the (ϵ, δ) definition to prove $\lim_{x \to 3} (2x^2 + 1) = 19$.

(c) $|f(x) - L| = |2x^2 + 1 - 19| = |2x^2 - 18| = 2|x-3| \cdot |x+3|$

Let $\varepsilon > 0$.

Preliminary choice. Let $\delta \le 1$.

if $|x-3| < \delta \le 1 \Rightarrow |x-3| < 1 \Rightarrow -1 < x-3 < 1 \overset{\text{add } 6}{\Rightarrow} 5 < x+3 < 7$

So $\boxed{\text{if } |x-3| < \delta \le 1 \text{ then } |x+3| < 7}$ (*)

Final choice: let $\underline{\delta = \min\left(1, \frac{\varepsilon}{14}\right)}$ (so $\delta \le 1$ and $\delta \le \frac{\varepsilon}{14}$)

Then if $|x-3| < \delta$ then (*)

$|f(x) - L| = 2|x-3| \cdot |x+3| \overset{(*)}{<} 2 \cdot |x-3| \cdot 7 < 14 \cdot \delta \le \cancel{14} \cdot \frac{\varepsilon}{\cancel{14}}$

9. (10 pts) Choose ONE of the following:

(a) Assuming the inequality $\sin x \leq x \leq \tan x$ for any $x \in [0, \pi/2)$, prove that $\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$.

(b) State and prove the Junk Rule Theorem for rational functions.

See notes