NAME: Solution Key Panther ID: ____________

Exam 2 - MAC 2311 Summer B 2017

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (8 pts) The function $h(x)$ is given by $h(x) = \frac{x^2}{f(x)}$. It is known that $f(2) = 3$ and $f'(2) = 2$. Compute

(a) (2pts) $h(2) = \frac{2^2}{f(2)}$

$h(2) = \boxed{\frac{4}{3}}$

(b) (6pts) $h'(2)$

$h'(x) = \left(\frac{x^2}{f(x)}\right)' = \frac{2xf(x) - x^2 f'(x)}{(f(x))^2}$ (by Q. Rule)

$h'(2) = \frac{2\cdot 2\cdot f(2) - 2^2\cdot f'(2)}{(f(2))^2}$

$h'(2) = \frac{2\cdot 2\cdot 3 - 2^2\cdot 2}{3^2} = \boxed{\frac{4}{9}}$

2. (10 pts) Show that $y = e^{-x^2}$ satisfies the differential equation $y'' + 2xy' + ay = 0$ for a certain constant a that you should determine.

1.1

$y' = \left(e^{-x^2}\right)' = -2xe^{-x^2}$

$y'' = \left(-2xe^{-x^2}\right)' \underset{\text{prod. rule}}{=} -2e^{-x^2} - 2xe^{-x^2}\cdot(-2x)$

$y'' = -2e^{-x^2} + 4x^2e^{-x^2}$

Thus $y'' + 2xy' + ay =$

$= -2e^{-x^2} + \cancel{4x^2e^{-x^2}} - \cancel{4x^2e^{-x^2}} + a\cdot e^{-x^2}$

This is equal to 0 when $a = 2$

Thus $y = e^{-x^2}$ is a solution for the diff. eq. $y'' + 2xy' + 2y = 0$

Sol. 2 - use log. differentiation

$\ln y = \ln\left(e^{-x^2}\right) = -x^2$

Differentiate both sides.

$\frac{1}{y}\cdot y' = -2x \Rightarrow$

$y' = -2xy$ (Apply $\frac{d}{dx}$ again)

$y'' = (-2xy)' = -2y - 2xy'$

Thus $y'' + 2xy' + 2y = 0$, so given y is a solution when $a = 2$.

3. (30 pts) Find the derivative of each of the following functions. Simplify your answer when possible (6 pts each):

(a) $y = 5x^4 - \frac{4}{x} + \frac{e^3}{3}$ $= 5x^4 - 4x^{-1} + \text{const}$ $\uparrow \frac{e^3}{3}$

$y' = 20x^3 + 4x^{-2}$

$\boxed{y' = 20x^3 + \frac{4}{x^2}}$

(b) $y = x\arctan(x^2)$

$y' = (x\arctan(x^2))'$ prod rule

$= 1\cdot\arctan(x^2) + x\cdot\underbrace{\frac{1}{1+(x^2)^2}\cdot 2x}_{\text{Chain Rule}}$

$\boxed{= \arctan(x^2) + \frac{2x^2}{1+x^4}}$

(c) $y = \ln(\sec x + \tan x)$

$y' = \frac{1}{\sec x + \tan x}\cdot(\sec x\cdot\tan x + \sec^2 x)$

$y' = \frac{\sec x\,(\cancel{\tan x + \sec x})}{\cancel{\sec x + \tan x}} = \boxed{\sec x}$

(d) $y = \sqrt{1+\sin^2(3x)}$ $= (1+(\sin(3x))^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}\left(1+(\sin(3x))^2\right)^{-\frac{1}{2}}\cdot\left(1+(\sin(3x))^2\right)'$

$y' = \frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{1+\sin^2(3x)}}\cdot\cancel{2}\sin(3x)\cdot\cos(3x)\cdot 3$

$\boxed{y' = \frac{3\sin(3x)\cos(3x)}{\sqrt{1+\sin^2(3x)}}}$

(e) $y = x^{\ln x}$

Use log. differentiation

$\ln y = \ln(x^{\ln x}) = (\ln x).(\ln x) = (\ln x)^2$ | Apply $\frac{d}{dx}$

$(\ln y)' = ((\ln x)^2)'$

$\frac{1}{y}\cdot y' = 2(\ln x)\cdot\frac{1}{x}$

$\boxed{y' = \frac{2\ln x}{x}\cdot y = 2\frac{\ln x}{x}\cdot x^{\ln x}}$

3. (8 pts) These are True or False questions. Circle your answer. No justification necessary.

(a) If $y = \cos x$, then $y'' = -y$. **True** **False** $y' = -\sin x \quad y'' = -\cos x = -y$

(b) If $y = g(u)$ and $u = h(x)$, then $\frac{dy}{dx} = g'(u) \cdot h'(x)$. **True** **False** Chain Rule

(c) $\left(\frac{1}{\tan x}\right)' = \frac{1}{\sec^2 x}$ **True** **False** $\left(\frac{1}{\tan x}\right)' = (\cot x)' = -\csc^2 x \neq \frac{1}{\sec^2 x}$

(d) If $f(x)$ is continuous at $x = 3$ then $f(x)$ is differentiable at $x = 3$. **True** **False**

4. (14 pts) (a) (8 pts) Use implicit differentiation to find dy/dx for the curve $x^2 + xy + y^2 = 3$.

$$(x^2 + xy + y^2)' = (3)'$$

$$2x + 1 \cdot y + \underline{x \cdot y'} + \underline{2y \cdot y'} = 0$$

$$(x + 2y)y' = -y - 2x$$

$$\boxed{\frac{dy}{dx} = y' = \frac{-2x - y}{x + 2y}}$$

(b) (6 pts) Find the coordinates of the point(s) on the curve $x^2 + xy + y^2 = 3$ where the tangent line vertical?

Vertical tang. line corresponds to points where $\frac{dy}{dx}$ is not defined, so to points where $x + 2y = 0$

Use this together with the initial equation

$$\begin{cases} x + 2y = 0 & \Rightarrow x = -2y \\ x^2 + xy + y^2 = 3 \end{cases}$$

$$(-2y)^2 + (-2y) \cdot y + y^2 = 3$$

$$4y^2 - 2y^2 + y^2 = 3 \Rightarrow 3y^2 = 3 \Rightarrow y^2 = 1$$

$$\Downarrow$$

$$y = \pm 1$$

Thus we have two points where the tang. line is vertical

$\boxed{(x = -2,\ y = 1) \quad \text{and} \quad (x = 2,\ y = -1).}$

6. (10 pts) Use an appropriate local linear approximation to estimate cot(43°). Be sure to specify the function and the point you are using for the approximation.

Will use loc. lin. approx. for $f(x) = \cot x$ at $x_0 = \frac{\pi}{4}$

angle should be in radians

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0) = \cot\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = (\cot x)' = -\csc^2 x = -\frac{1}{\sin^2 x}$$

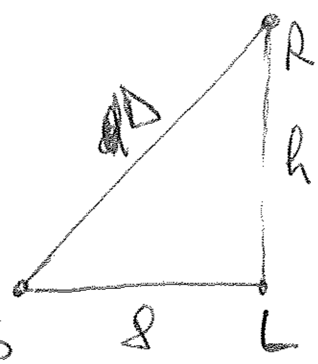
$$f'(x_0) = f'\left(\frac{\pi}{4}\right) = -\csc^2\frac{\pi}{4} = -\frac{1}{\sin^2\frac{\pi}{4}} = -2$$

So $\cot x \approx 1 - 2\left(x - \frac{\pi}{4}\right)$

$$\cot(43^\circ) = \cot\left(\frac{43\pi}{180}\right) \approx 1 - 2\left(\frac{43\pi}{180} - \frac{\pi}{4}\right)$$

or $\boxed{\cot(43^\circ) \approx 1 + \frac{\pi}{45}}$

7. (12 pts) A rocket that is launched vertically is tracked by a radar station located on the ground 8 miles from the launch site. What is the vertical speed of the rocket at the instant when its distance from the radar station is 10 miles and this distance increases at the rate of 2400 mi/h?



Let h ← height of rocket

D ← distance between rocket and radar station

Both h and D vary with time.

We need to find $\frac{dh}{dt} = ?$ when

$D = 10$ and $\frac{dD}{dt} = 2400 \frac{mi}{h}$

$$D^2 = 8^2 + h^2 \quad \Big| \text{ Take } \frac{d}{dt}$$

$$2D \cdot \frac{dD}{dt} = 0 + 2h \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{D \cdot \frac{dD}{dt}}{h}$$

when $D = 10 \Rightarrow h = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$

Thus $\frac{dh}{dt} = \frac{10 \cdot 2400}{6} = \boxed{4000 \frac{mi}{h}}$

This is the vertical speed of the rocket at that moment.

8. (20 pts) Choose TWO of the following three options (10 pts each). Indicate clearly the two parts you choose.

(A) Use the limit definition of the derivative to find, with proof, the formula for $\left(\sin x\right)'$.

(B) Find, with proof, the formula for $\left(\arccos x\right)'$.

(C) Find, with proof, a formula for $\left(f(g(x))\right)''$ in terms of the first and second derivatives of f and g.

For (A) & (B) see notes or textbook

Solution for (C)

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x) \quad \text{Chain Rule}$$

$$\left(f(g(x))\right)'' = \left(f'(g(x)) \cdot g'(x)\right)' \overset{\text{prod. rule}}{=}$$

$$= \left(f'(g(x))\right)' \cdot g'(x) + f'(g(x)) \cdot g''(x)$$

$$= \underbrace{f''(g(x)) \cdot g'(x)}_{\text{Chain Rule}} \cdot g'(x) + f'(g(x)) \cdot g''(x)$$

Thus

$$\left(f(g(x))\right)'' = f''(g(x)) \cdot \left(g'(x)\right)^2 + f'(g(x)) \cdot g''(x)$$