Exam 1 Spring 2013 Answers

1a) $v'(t)$ represents $\longrightarrow$ $\boxed{\text{acceleration}}$

$\int_0^{10} |v(t)|\,dt$ represents $\longrightarrow$ $\boxed{\text{Total distance travelled}}$

b) The integral $\int_2^3 r(t)\,dt$ represents $\longrightarrow$ oil spilt on day 3.

c) Simplify as much as possible the expression

$$\frac{d}{dx}\left(\int_e^{e^x} (\ln(t))^2\,dt\right) = \ln(e^x)^2 \cdot e^x \Rightarrow \boxed{x^2 \cdot e^x}$$

2) a) Find area under graph of $f(x) = \cos x$ over interval $[0, \pi/2]$

$$\int_0^{\pi/2} \cos x\,dx = \sin x \Big|_{x=0}^{x=\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = \boxed{1}$$

b) Average value of $f(x) = \cos x$, where $x \in [0, \pi/2]$

$$= \frac{\int_0^{\pi/2} \cos x\,dx}{\pi/2 - 0} = \frac{2}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}\left[\sin x\right]\Big|_{x=0}^{x=\pi/2} = \boxed{\frac{2}{\pi}}$$

$\frac{2}{\pi} \approx 0.64$, so, $\frac{2}{\pi} > \frac{1}{2}$.

3) Stone dropped from building 192 ft. tall. How long until it hits the ground? (Where $g = 32$ ft/s^2)

$a = -32$ $\quad s_0 = 192$ $\quad s(t) = s_0 + v_0(t) + \frac{1}{2}at^2$

$v_0 = 0$

$s(t) = 192 - 16t^2$, $0 = 192 - 16t^2 \Rightarrow 16t^2 = 192$

$t^2 = 192/16 \Rightarrow t = \sqrt{\frac{192}{16}}$

$\boxed{t = 2\sqrt{3}}$

4a) $\int_1^4 \left(\frac{2}{x^2} + \frac{1}{\sqrt{x}}\right) dx$

$= \int_1^4 (2\cdot x^{-2} + x^{-1/2})\,dx = 2\int_1^4 x^{-2}\,dx + \int_1^4 x^{-1/2}\,dx$

$= 2\,\frac{x^{-1}}{-1}\Big|_{x=1}^{x=4} + \frac{2}{1}\cdot x^{1/2}\Big|_{x=1}^{x=4} = -\frac{2}{x}\Big|_{x=1}^{x=4} + 2\sqrt{x}\Big|_{x=1}^{x=4}$

$= \left(-\frac{2}{4} + \frac{2}{1}\right) + \left(2\sqrt{4} - 2\sqrt{1}\right) = \left(-\frac{1}{2} + 2\right) + (4 - 2) = \frac{3}{2} + 2 = \boxed{\frac{7}{2}}$

b) $\int_0^3 \sqrt{6x - x^2}\,dx \longrightarrow -x^2 + 6x + b^2 = 3^2 = (-x+3)^2 = 3^2$

$\Rightarrow 3^2 = -(x-3)^2 \Rightarrow 3^2 - (x-3)^2 = 0$

$= \int_0^3 \sqrt{3^2 - (x-3)^2}\,dx \qquad A = \frac{\pi r^2}{4} \Rightarrow \frac{\pi \cdot (3)^2}{4}$

$\boxed{A = \frac{9\pi}{4}}$

c) $\displaystyle\int_{x=0}^{x=\pi/4} 4\sin(2x)(1+\cos(2x))^3\,dx$

$w = 1+\cos(2x)$

$dw = -\sin(2x)\cdot 2\,dx$

$\Rightarrow -\frac{1}{2}dw = \sin 2x\,dx$

$$= 4\int_{x=0}^{x=\pi/4} \sin(2x)(1+\cos(2x))^3\,dx \Rightarrow -\frac{4}{2}\int_{w=2}^{w=1} w^3\,dw = 2\int_{w=1}^{w=2} w^3\,dw$$

$$= 2\cdot\frac{1}{4}w^4\Big|_{w=1}^{w=2} \Rightarrow \frac{1}{2}\left[16-1\right] = \frac{1}{2}\cdot 15 = \boxed{\frac{15}{2}}$$

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d) $\displaystyle\int_{x=-\ln 2}^{x=\ln 2} \frac{e^x}{e^x+3}\,dx$

$w = e^x+3$

$dw = e^x\,dx$

$w = e^{\ln 2}+3 = \boxed{5}$

$w = e^{-\ln 2}+3 =$ [illegible]

$$\Rightarrow \int_{w=1}^{w=5} \frac{1}{w}\cdot dw \Rightarrow \ln(w)\Big|_{w=1}^{w=5} \Rightarrow (\ln 5) - \ln(1) = \boxed{\ln 5}$$

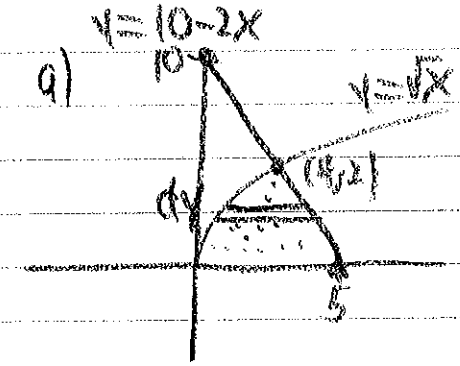
5) Find Value of the sum: $2+4+6+8+\ldots+998+1000$

$$= \sum_{k=1}^{500}(2k) = 2\sum_{k=1}^{500} k = \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} = n(n+1)$$

where $n = 500$, $\Rightarrow \boxed{500(501)}$

6) Area bounded by $y=\sqrt{x}$, $y=10-2x$, $y=0$.

a)



$y=10-2x$

$2x=10-y$

x-line $\rightarrow \boxed{x=5-\frac{y}{2}}$ $x_{par.} \rightarrow \boxed{x=y^2}$

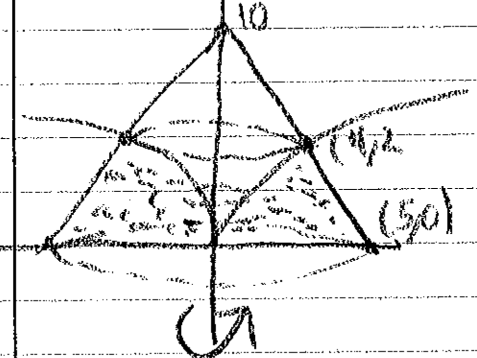
Th = dy

$A = x_{line} - x_{parabola}$

b) $A=\int_0^2 \left(\left(5-\frac{y}{2}\right)-y^2\right)dy \Rightarrow A = 5y - \frac{y^2}{4} - \frac{y^3}{3}\Big|_{y=0}^{y=2}$

$A = \left(10-1-\frac{8}{3}\right) = \left(9-\frac{8}{3}\right) = \frac{27}{3}-\frac{8}{3} = \boxed{\frac{19}{3}}$

c)



Use discs. $A=\pi(R^2-r^2)$

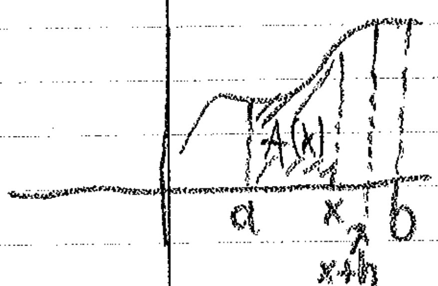
Th = dy $R = x_{line} = 5-\frac{y}{2}$

$r = x_{curve} = y^2$

$$\boxed{V=\int_{y=0}^{y=2}\pi\left[\left(5-\frac{y}{2}\right)^2-\left(y^2\right)^2\right]dy}$$

7) Prove FTC part II

- If $f(t)$ is continuous when $t \in [a, b]$



$$A(x) = \int_a^x f(t)\,dt \Rightarrow \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

- $$\frac{d}{dx}A(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

By MVT, $\lim_{h \to 0} \frac{f(x^*) \cdot \cancel{h}}{\cancel{h}}$, as $h \to 0 \Rightarrow x^* \to x$

so $\lim_{h \to 0} \frac{f(x^*) \cdot \cancel{h}}{\cancel{h}} = f(x)$