The Fundamental Theorem of Calculus:
Assume that $f(x)$ is a continuous function on an interval $[a, b]$.
(i) If $F(x)$ is an anti-derivative for $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)\ .$$

(ii) The net-signed area function $A(x) = \int_a^x f(t)\, dt$ is an anti-derivative for f on the interval. That is

$$\frac{d}{dx}\Big(\int_a^x f(t)\, dt\Big) = f(x)\ .$$

Proof: We start with part (ii). We use the limit definition of the derivative to show $A'(x) = f(x)$.

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\, dt}{h}$$

For the last equality we used the additive property of integral on adjacent intervals:

$$\int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt = \int_a^{x+h} f(t)\, dt\ .$$

From the intermediate value theorem for the integral of $f(t)$ on the interval $[x, x+h]$, it follows that there is a point $x^* \in [x, x+h]$ so that

$$\int_x^{x+h} f(t)\, dt = f(x^*) \cdot h\ \ .$$

Thus, we get

$$A'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\, dt}{h} = \lim_{h\to 0} \frac{f(x^*) \cdot h}{h} = \lim_{h\to 0} f(x^*)\ .$$

But now remember that x^* is trapped in the interval $[x, x+h]$. So when $h \to 0$, x^* must approach x. Since the function f is continuous, we have $\lim_{h\to 0} f(x^*) = f(x)$. Hence we proved $A'(x) = f(x)$. QED

Next, here is the simple proof of part (i) using part (ii). Let $F(x)$ be an anti-derivative of $f(x)$. We proved in (ii) that $A(x)$ is also an anti-derivative of $f(x)$, thus $F'(x) = A'(x) = f(x)$. It follows that $(F(x) - A(x))' = F'(x) - A'(x) = f(x) - f(x) = 0$ on the entire interval, so $F(x) - A(x) = c$, where c is a constant. Write this as $F(x) = A(x) + c$. From this,

$$F(b) - F(a) = (A(b) + c) - (A(a) + c) = A(b) - A(a)\ .$$

But from the definition of $A(x)$, we get

$$A(a) = \int_a^a f(t)\, dt = 0 \text{ and } A(b) = \int_a^b f(t)\, dt\ .$$

Thus, we proved that

$$F(b) - F(a) = \int_a^b f(t)\, dt\ . \text{ QED}$$