NAME: Solution Key   Panther ID: ____________

Exam 2 - MAC 2313   Fall 2018

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (15 pts) Given the function $f(x,y) = \ln(1+x^2+3y^2)$ find:

(a) (5 pts) The partial derivatives f_x, f_y at an arbitrary point (x,y).

$$f_x = \frac{2x}{1+x^2+3y^2} \qquad f_y = \frac{6y}{1+x^2+3y^2}$$

Note: The intended function for this problem was $f(x,y) = \ln(1+x+3y^2)$. Try with this function parts (b) and (c).

(b) (5 pts) The directional derivative of f at $P(0,0)$ in the direction of $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$ (note that $\mathbf{a}$ is not a unit vector).

$$(D_{\vec{u}} f)(0,0) = (\nabla f)(0,0) \cdot \vec{u} \quad \text{where } \vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{4\vec{i}-3\vec{j}}{\sqrt{4^2+3^2}} = \frac{1}{5}(4\vec{i}-3\vec{j})$$

$$(\nabla f)(0,0) = f_x(0,0)\vec{i} + f_y(0,0)\vec{j} = 0\vec{i} + 0\vec{j} = \vec{0}$$

so $(D_{\vec{u}} f)(0,0) = 0$ ← so the instantaneous rate of change of the function in that direction at $(0,0)$ is zero

(c) (5 pts) A unit vector in the direction in which f increases most rapidly at $P(0,0)$ and the rate of increase in this direction.

~~As observed above~~ Normally, you'd say that the gradient of the function at the point gives the direction of most rapid increase. But this is true at points where the gradient is a non-zero vector.

As observed above $(\nabla f)(0,0) = \vec{0}$, so in this case with respect to any direction $\vec{u}$ the ~~rate of~~ directional derivative $(D_{\vec{u}} f)(0,0) = 0$. So with respect to all directions the rate of change of the function at $(0,0)$ is 0.

But $(0,0)$ is a relative minimum point (show that!) and if you look at the second partial derivatives at $(0,0)$, you'll find that in the direction of y axis $(\vec{j})$ you find the biggest concavity, so that is the direction for most rapid increase in this case.

2. (20 pts) True or False questions. Circle your answer and give a brief justification (4 pts each).

(a) For any moving particle, the velocity vector and the unit tangent vector are parallel. **True** **False**

Justification: $\vec{v}(t) = \vec{r}\,'(t) = \|\vec{r}\,'(t)\| \cdot \vec{T}(t)$

(b) If $\frac{ds}{dt} = 3$ for all t, then $\mathbf{r}'(t) \perp \mathbf{r}''(t)$ for all t. **True** **False**

Justification: $\frac{ds}{dt} = \|\vec{r}\,'(t)\| = 3$ ← constant so $\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$ by a theorem whose proof you had to study.

Alternative justifications: $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{3}\vec{r}\,'(t)$ $\quad \vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{\frac{1}{3}\vec{r}\,''(t)}{\frac{1}{3}\|\vec{r}\,''(t)\|}$ So, in this case, $\vec{r}\,''(t)$ is colinear with $\vec{N}(t)$. But $\vec{T}(t) \perp \vec{N}(t)$ so $\vec{r}\,'(t) \perp \vec{r}\,''(t)$

(c) If $z = z(x, y)$ and $x = r\cos\theta, y = r\sin\theta$ then $\frac{\partial z}{\partial \theta} = -r\sin\theta + r\cos\theta$ **True** **False**

Justification: By chain rule

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

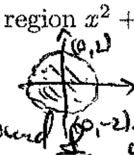
← for a general function z, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are not 1.

(d) If $D = f_{xx}f_{yy} - (f_{xy})^2 < 0$ at a critical point P, then f has a relative extremum at P. **True** **False**

Justification: As $\Delta < 0$, P is a saddle point so it is not a rel. min or a rel. max. for the function.

(e) The function $f(x, y) = y^3$ has no absolute maximum on the region $x^2 + y^2 < 4$. **True** **False**

Justification: The region is the open disk



On this region $y^3 < 8$, but the upper bound is never reached as the point $(0,2)$ is not in the region

3. (10 pts) Suppose that $p(x, y)$ denotes the atmospheric pressure at a point (x, y). Given that $p(100, 98) = 1008$ mb (millibars), $p_x(100, 98) = -2$ mb/km and $p_y(100, 98) = 1$ mb/km, use local linear approximation to estimate the atmospheric pressure at the point $(103, 100)$.

Local linear approximation

$$\Delta p \approx p_x(x_0, y_0)\Delta x + p_y(x_0, y_0)\Delta y \quad \text{or}$$

$$p(x,y) \approx p(x_0, y_0) + p_x(x_0, y_0)(x - x_0) + p_y(x_0, y_0)(y - y_0)$$

so $p(\underset{x}{103}, \underset{y}{100}) \approx p(\underset{x_0}{100}, \underset{y_0}{98}) + p_x(100, 98)(103 - 100) + p_y(100, 98)(100 - 98)$

$$p(103, 100) \approx 1008 - 2(3) + 1 \cdot (2)$$

so $p(103, 100) \approx \underline{1004 \text{ mb}}$.

4. (10 pts) Find the equation of the tangent plane to the ellipsoid $\underbrace{x^2 + 4y^2 + z^2}_{\overset{\shortparallel}{F(x,y,z)}} = 18$ at the point $(1,\ 2,\ 1)$.

Normal $\vec{n} = (\nabla F)(1,2,1)$

$(\nabla F)(x,y,z) = \langle 2x, 8y, 2z \rangle$ so $\vec{n} = (\nabla F)(1,2,1) = \langle 2, 16, 2 \rangle$

Tangent plane is

$$\boxed{2\cdot(x-1) + 16(y-2) + 2(z-1) = 0}$$ or

$$(x-1) + 8(y-2) + (z-1) = 0$$ or

$$x + 8y + z = 18$$

5. (10 pts) Find the curvature $\kappa(t)$ of the ellipse $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, for $t \in [0, 2\pi]$. Use the formula

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$\vec{r}\,'(t) = -3\sin t\,\vec{i} + 2\cos t\,\vec{j}$

$\vec{r}\,''(t) = -3\cos t\,\vec{i} - 2\sin t\,\vec{j}$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = (6\sin^2 t + 6\cos^2 t)\,\vec{k} = 6\vec{k}$

so $$\kappa(t) = \frac{\|6\vec{k}\|}{\|-3\sin t\,\vec{i} + 2\cos t\,\vec{j}\|^3} = \boxed{\frac{6}{\left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3}}$$

or $$\kappa(t) = \frac{6}{\sqrt{4 + 5\sin^2 t}} \Bigg|, \quad t \in [0, 2\pi].$$

6. (10 pts) If the following limit exists, compute it. If the limit does not exist, justify why it doesn't.

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}}$$

Good to treat this in polar coordinates

$$\frac{x+y}{\sqrt{x^2+y^2}} = \frac{r\cos\theta + r\sin\theta}{r} = \cos\theta + \sin\theta$$

But as $(x,y)\to(0,0)$, equivalently, if $r\to 0_+$, the expression on the right depends on θ. Thus if we approach the origin on rays with different θ, we'd get different values of the limit

Thus, $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}}$ does not exist.

7. (15 pts) At what point(s) on the circle $x^2 + y^2 = 1$ does the function $f(x,y) = xy$ have an absolute maximum value and what is that max ? Lagrange multipliers method is suggested, but a parametrization should also work.

Solution with Lagrange multipliers: We are looking to maximize $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2+y^2 = 1$

Critical pts $\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2=1 \end{cases} \iff \begin{cases} \langle y, x\rangle = \lambda\langle 2x, 2y\rangle \\ x^2+y^2=1 \end{cases} \iff$

$\iff \begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2+y^2=1 \end{cases}$ substitution $x = 2\lambda(2\lambda x)$ so $x = 4\lambda^2 \cdot x \Rightarrow x(1-4\lambda^2) = 0$

If $x=0$, by $y = 2\lambda x$ we get also $y=0$, but the 3rd equation is not satisfied.

So $1-4\lambda^2 = 0 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$.

This means that $y = \pm x$ and using the 3rd equation we get four critical pts $\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right), \left(-\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right), \left(\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right), \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

Evaluating $f(x,y) = xy$ at these points, it's clear that the maximum occurs at the first two and the maximum value of the function is $\frac{1}{2}$.

8. (15 pts) Choose ONE proof. If you do both, only the larger score will be considered for this problem, but the second proof may give some bonus towards a previous problem where your score is smaller. You can use the back of the page.

(A) Find (with proof) the parametric equations of the projectile motion.

(B) Prove that for a differentiable function $f(x, y)$, the gradient is normal to the level curves of f.

See notes or textbook