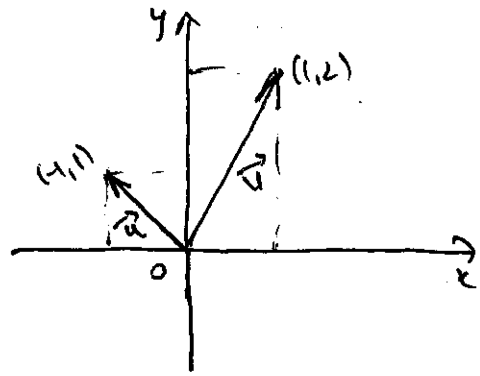
Name: Solution Key Panther ID: ____________

Exam 1 MAC-2313 Fall 2018

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (20 pts) Given the vectors $\mathbf{u} = -\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, do the the following (5 pts each):

(a) Sketch $\mathbf{u}$ and $\mathbf{v}$ as vectors in the xy-plane with initial point at the origin.



(b) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.
(If you get an answer like $\theta = \arcsin(1/5)$, for example, you do not have to simplify).

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$\text{so } \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{-1+2}{\sqrt{2}\cdot\sqrt{5}} = \frac{1}{\sqrt{10}} \Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{10}}\right)$$

(c) Find the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

$$\text{Area} = \|\vec{u} \times \vec{v}\|$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -3\vec{k} \qquad \text{so Area} = \|-3\vec{k}\| = \boxed{3}$$

(d) Find a vector $\mathbf{w}$ with length $\sqrt{17}$ and with the same direction as $\mathbf{u}$.

$$\vec{w} = \sqrt{17}\cdot\frac{\vec{u}}{\|\vec{u}\|} = \frac{\sqrt{17}}{\sqrt{2}}\langle -1, 1\rangle = \sqrt{\frac{17}{2}}\left(-\vec{i}+\vec{j}\right)$$

2. (14 pts) Circle True or False. You do not have to explain these. Assume that $\mathbf{u}, \mathbf{v}$ are arbitrary vectors in $\mathbf{R}^3$ unless stated otherwise.

(a) If $\mathbf{u} \perp \mathbf{v}$, then $(5\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + 2\mathbf{v}) = 5\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$. **True** **False**

$\vec{u} \cdot \vec{v} = 0$ (since $\vec{u} \perp \vec{v}$)
$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$, etc.

(b) For any non-zero vectors $\mathbf{u}, \mathbf{v}$, $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$. **True** **False**

(c) Every plane has exactly two unit normal vectors. **True** **False**

(d) If two planes intersect in a line L, then L is parallel to the cross product of the normals to the two planes. **True** **False**

(e) $z = 3x^2 + y^2$ is a parabolic hyperboloid. **True** **False**

It's an elliptic paraboloid.
There is no such thing as parabolic hyperboloid

(f) $3y^2 - z^2 = 1$ is a hyperbolic cylinder. **True** **False**

(g) The graph of $\mathbf{r}(t) = (3 - 2t)\mathbf{i} + 5t\mathbf{j} + (1 - t)\mathbf{k}$ is a line in 3-space. **True** **False**

3. (10 pts) The lines L_1 and L_2 are given by the following parametric equations:

$$L_1: \ x = 1 + 7t, \ y = 3 + t, \ z = 5 - 3t, \qquad L_2: \ x = 4 - s, \ y = 6, \ z = 7 + 2s \ .$$

Determine if the the lines L_1, L_2 are parallel, intersect, or are skew. Justify your answer.

To find a (potential) common point, we should solve the system

$$\begin{cases} 1 + 7t = 4 - s \\ 3 + t = 6 \\ 5 - 3t = 7 + 2s \end{cases}$$

From the first two equations we get $t = 3$ and $s = -18$.
But the third equation is not satisfied, as $5 - 3 \cdot 3 \neq 7 + 2 \cdot (-18)$)
Thus, the system has no solution, so the lines do not intersect

Moreover, as the directional vectors
$\vec{u}_1 = \langle 7, 1, -3 \rangle$ and $\vec{u}_2 = \langle -1, 0, 2 \rangle$ are not scalar multiples of one another, we conclude $L_1 \nparallel L_2$.
Thus L_1 and L_2 are skew.

4. (10 pts) Find an equation of the plane that contains the origin $O(0,0,0)$ and the line $x = 1 + 7t,\ y = 3 + t,\ z = 5 - 3t$.

If $t=0$, we get the point $P(1,3,5)$ on the line
The directional vector $\vec{u} = \langle 7, 1, -3 \rangle$ of the line and the vector $\vec{OP} = \langle 1,3,5 \rangle$ are vectors in the plane, so
$\vec{n} = \vec{u} \times \vec{OP}$ is normal to the plane

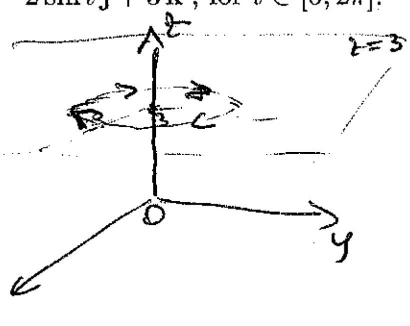
$$\vec{n} = \vec{u} \times \vec{OP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 1 & -3 \\ 1 & 3 & 5 \end{vmatrix} = 14\vec{i} - (38)\vec{j} + 20\vec{k} = 14\vec{i} - 38\vec{j} + 20\vec{k}$$

Thus, an equation for the plane is $14(x-0) - 38(y-0) + 20(z-0) = 0$

or $\underline{14x - 38y + 20z = 0}$

5. (12 pts) Consider the vector-valued function $\mathbf{r}(t) = 2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} + 3\,\mathbf{k}$, for $t \in [0, 2\pi]$.

(a) (6 pts) Sketch a graph of $\mathbf{r}(t)$ in 3d and briefly describe the shape in words.



It's a circle of radius 2 in the plane $z=3$ with center at $(0,0,3)$, traced clock-wise.

(b) (6 pts) Compute $\|\mathbf{r}'(t)\|$.

$$\vec{r}\,'(t) = -2\sin t\,\vec{i} - 2\cos t\,\vec{j} + 0\cdot\vec{k}$$

so $$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 0} = \sqrt{4} = \boxed{2}$$

6. (10 pts) Find the point of intersection (if any) of the line $x = 1 + t, y = 1 - t, z = 2t$ with the plane $x + y + z = 4$.

We should solve the system

$$\begin{cases} x = 1+t \\ y = 1-t \\ z = 2t \\ x+y+z = 4 \end{cases}$$

Substituting first 3 in the 4th, we get

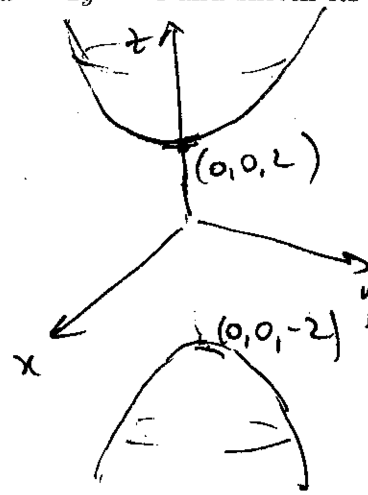
$$(1+t) + (1-t) + 2t = 4, \text{ so } 2t = 2, \text{ so } t = 1$$

Thus, the line intersects the plane at the point

$P(1+1, 1-1, 2\cdot 1)$ or $\underline{P(2,0,2)}$.

7. (12 pts) (a) (6 pts) Specify the type of the quadric surface $z^2 - x^2 - 2y^2 = 4$ and sketch its graph (part (b) might also help).

It's a hyperboloid with 2 sheets



(b) (6 pts) What is the intersection of the surface $z^2 - x^2 - 2y^2 = 4$ with the plane $z = 3$? What about the intersection with the plane $z = 1$?

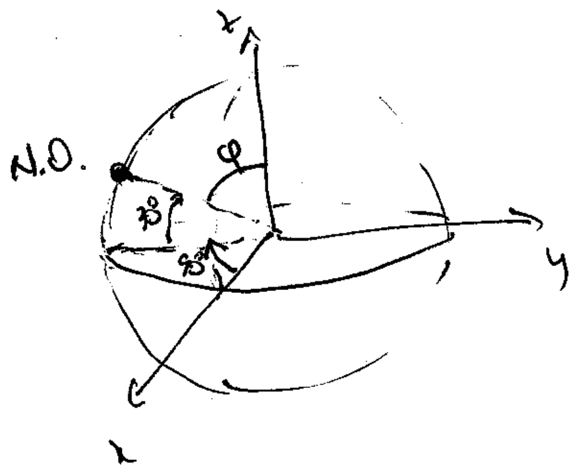
The trace of $z^2 - x^2 - 2y^2 = 4$ in the $z = 3$ plane is the curve

$$\begin{cases} z^2 - x^2 - 2y^2 = 4 \\ z = 3 \end{cases} \text{ or } \begin{cases} 3^2 - x^2 - 2y^2 = 4 \\ z = 3 \end{cases} \text{ or } \begin{cases} -x^2 - 2y^2 = -5 \\ z = 3 \end{cases}$$

or $\begin{cases} x^2 + 2y^2 = 5 \\ z = 3 \end{cases}$ $\Rightarrow$ it's an ellipse in the plane $z = 3$

For $z = 1$, get $+x^2 + 2y^2 = -3$, so the intersection of the surface with $z = 1$ is the empty set

8. (10 pts) Find the rectangular coordinates (x, y, z) of New Orleans given that its geographical coordinates are 90° West longitude and 30° North latitude. Assume the Earth is a sphere of radius 4000 miles. Assume also that the coordinate system is chosen so that the origin is at the center of the Earth, the xy plane corresponds to the plane of the equator and the xz-plane corresponds to the prime meridian (which also contains Greenwich, England).



The spherical coordinates (ρ, θ, φ) of New Orleans are $\rho = 4000$

$\theta = -90°$ (West of prime meridian correspond to negative θ)

or

$\theta = 360° - 90° = 270°$

or $\theta = \frac{3\pi}{2}$

$\varphi = 90° - 30° = 60°$

or $\varphi = \frac{\pi}{3}$

The rectangular coordinates of N.O. will be then

$$\begin{cases} x = \rho \sin\varphi \cos\theta = 4000 \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi}{2}\right) = 0 \\ y = \rho \sin\varphi \sin\theta = 4000 \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{2}\right) = -4000\frac{\sqrt{3}}{2} = -2000\sqrt{3} \\ z = \rho \cos\varphi = 4000 \cos\left(\frac{\pi}{3}\right) = 4000 \cdot \frac{1}{2} = 2000 \end{cases}$$

Thus $(x = 0, \ y = -2000\sqrt{3}, \ z = 2000)$

are the rectangular coords. of New Orleans (with respect to coordinate axis chosen as described in the problem)

9. (12 pts) Choose ONE proof. If you do both, only the larger score will be considered for this problem, but the second proof may give some bonus towards a previous problem where your score is smaller. You can use the back of the page.

(A) Prove the point-normal equation of a plane. That is, you should find (with proof) the equation of a plane in R^3 through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = < a, b, c >$.

(B) Prove Theorem 11.4.6a, that $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. If you like, you can assume $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$ for simplicity.

See the notes or text book.