NAME: ______________________________ **Panther ID:** ____________

Final Exam - MAC 2313 Spring 2007

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (27 pts) Set up iterated double or triple integrals to represent each of the following. Don't spend time trying to evaluate the integrals. It is not required. (Picture is required in each case).

(a) (9 pts) The area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = 1$.

(b) (9 pts) The volume of the wedge in the first octant cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$.

(c) (9 pts) The surface area of the part of the paraboloid $z = 16 - x^2 - y^2$ that lies above the xy-plane.

2. (20 pts) Write the defining formula for each of the following. Make sure to explain the notations you are using in each case.

(a) Gradient of a function $f(x, y, z)$;

(b) Curl of a vector field $\mathbf{F}$;

(c) The directional derivative $D_{\mathbf{u}}f$ of a function f with respect to a unit vector $\mathbf{u}$.

(d) Unit tangent vector to a curve $\mathbf{r}(t)$.

(e) The equation of the tangent plane at a point (x_0, y_0, z_0) of a surface given by $G(x, y, z) = 0$, where $G(x, y, z)$ is a function with continuous first-order partial derivatives.

3. (18 pts) Given the points $A(1, 0, 1), B(0, 2, 3), C(1, 1, -1)$ in $\mathbf{R}^3$, find the following:

(a) (5 pts) The angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) (5 pts) The area of the triangle ABC.

(b) (8 pts) The equation of the plane that contains the points A, B, C.

4. (18 pts) Find the coordinates of the centroid of the hemisphere $x^2 + y^2 + z^2 = R^2$, $z \geq 0$. Assume that the density is constant, $\delta \equiv 1$. You may use the symmetry of the region to simplify your computations.

5. (18 pts) Find the work done by the vector field $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - \sin y)\mathbf{j}$ on a particle that moves along the line segment from $(0, 1)$ to $(1, 0)$.

6. (15 pts) Compute the integral

$$\int_C 3xy\ dx + 2x^2\ dy\ ,$$

where C is the counter-clock-wise oriented boundary of the region R shown in the picture. R is bounded above by the line $y = x$ and below by the parabola $y = x^2 - 2x$.

7. (18 pts) Let R be the the region enclosed in the first quadrant by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$ and $xy = 4$. Compute the area of the region R using the change of variables $u = xy$, $v = \frac{y}{x}$.

8. (15 pts) (a) (6 pts) Consider the sphere σ_a given by $x^2 + y^2 + z^2 = a^2$. Show that the unit outer-normal to a point (x, y, z) of the sphere σ_a is given by $\mathbf{n} = \mathbf{r}/a$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is, as usual, the position vector of the point (x, y, z).

(b) (9 pts) Consider an inverse square-field in 3-space

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\,\mathbf{r} = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Show that div $\mathbf{F} = 0$.

9. (16 pts) (a) (8 pts) With the notations from problem 8, find the outer flux of the inverse-square field $\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$ through the sphere σ_a, given by $x^2 + y^2 + z^2 = a^2$.

(*Hint:* Note that the flux-divergence theorem cannot be applied since $\mathbf{F}$ is not defined at the origin. However, using problem 8 and the definition of the flux, you can obtain the result with very little effort. You can use that the surface area of the sphere σ_a is $4\pi a^2$.)

(b) (8 pts) Prove Gauss Law for inverse-square fields: If $\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$ is an inverse square-field in 3-space and if σ is a closed oriented surface that surrounds the origin, then the outward flux of $\mathbf{F}$ through σ is $4\pi c$.