Name: ____________________________ **Panther ID:** ______________

Take home part of the Final Exam MAT 3501 Fall 2018

This is the take home part of the final exam, due Tuesday, Dec. 4. You are encouraged to collaborate, ask questions, but each of you should understand the solutions that you are handing in. You should also acknowledge collaborations or outside help.

1. Consider the triangle with sides $a = 5, b = 5, c = 6$. Find:

(a) The area of the triangle.

(b) The angles of the triangles (ok to leave them as inverse trig functions).

(c) The inradius r and the circumradius R of the triangle.

(d) The given triangle is an example of a *Heronian triangle*, i.e. a triangle whose sides are integers and the area is also an integer. Is the following statement true or false: "For any Heronian triangle, the inradius, r, and circumradius, R are rational numbers"? Justify your answer.

2. (a) Let $\triangle ABC$, whose sides we denote, as usual, a, b, c (that is, a is the length of the side opposite to A and so on). Let M be the midpoint of BC and denote by m_a the length of the median from A (that is, $m_a = |AM|$). Use the Law of Cosines in triangles $\triangle AMB$ and $\triangle AMB$, to obtain a formula for m_a only in terms of a, b, c. Write also the corresponding formulas for m_b and m_c.

(b) At the beginning of the 20th century it was conjectured that there exist no Heronian triangles for which two of the medians have rational lengths. This was shown to be false towards the end of the 20th century by Buchholz and Rathbun who discovered counterexamples. Show that the triangle $a = 73, b = 51, c = 26$ is one such counterexample.

3. Water is stored in a cone-shaped reservoir (vertex down) open at the top. Assuming the water evaporates at a rate proportional to the surface area exposed to the air, show that the depth of the water will decrease at a constant rate that does not depend on the dimensions of the reservoir.

4. (a) How many times in 24 hours are the two arms of a clock perpendicular to each other? Assume continuous motion for both arms.

(b) Find the exact times between 4pm and 5pm, when the two arms of a clock are perpendicular.

(c) Consider now a clock with three arms (one for hours, one for minutes, and one for seconds) with continuous motion for all three arms. Find, if possible, a moment when the hours and the minutes arms form an 180° angle and the seconds arm is perpendicular to the other arms. Assume again continuous motion for the arms.

5. (Adapted from UTeach materials) The Thrill Ride Company wants to build a roller coaster subject to a set of constraints. You are told that the company has several engineers who could design a blueprint and build the track if could only give them a formula for a function whose graph will be the desired curve of the track. Hence your task is to define a function over the interval $[0, 15]$ whose graph satisfies the following constraints (each unit represents 10 meters).

1. The entrance onto the track is at the point $(0, 10)$ and the exit is at $(15, 0)$. There are just two local extrema, a minimum at $(4, 2)$ and a maximum at $(8, 8)$. (You do not have to consider designing the stairs leading to the entrance.)

2. The slope of the curve at the entrance and exit points must be zero in order to facilitate getting on and off the roller coaster car.

3. The curve must be smooth, meaning that the function must be differentiable over its entire domain.

The engineers also suggest the following two options to attack the problem:

Option A: Make the roller coaster from one piece. This means you should find, if possible, a polynomial function $y = f(x)$ of suitable degree that satisfies all of the conditions.

Option B: Make the roller-coaster from several pieces, but each piece should be parabolic. Thus in option B, you need to find, if possible, a piece-wise defined function to satisfy all of the constraints and in each piece to only have quadratic expressions.

Can you be the designer of the roller coaster?