Solution Key

Worksheet 10/15/19 - Trig. Subs. - MAC 2312 Group nr. ____ **NAMES:** ____

Trigonometric substitutions are generally useful when dealing with integrals involving expressions of the type $\sqrt{a^2-x^2}$, or $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$. The general idea of a trig. substitution is to use the basic trig. identities

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta\,,$$

to get rid of square-root in the integral.

General rules: (I wrote the first one, you should fill in the blanks for the other two)

- If the integrand involves $\sqrt{a^2-x^2}$, then the substitution $x = a\sin\theta$ will usually be useful because

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta\,.$$

- If the integrand involves $\sqrt{a^2+x^2}$, then the substitution $x = a \cdot \tan\theta$ will usually be useful because

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$

- If the integrand involves $\sqrt{x^2-a^2}$, then the substitution $x = a\sec\theta$ will usually be useful because

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$

Next, compute each of the integrals, by using the appropriate trigonometric substitution.

1. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

2. $\displaystyle\int \frac{dx}{(4x^2-1)^{3/2}}\,dx$

3. $\displaystyle\int_3^4 \frac{1}{\sqrt{x^2-6x+10}}\,dx$ *Hint:* First, complete the square, then use a trig sub.

4. (bonus, if you have time) Earlier in the course we mentioned that it is not easy to compute the integral

$\displaystyle\int \sqrt{a^2-x^2}\,dx$ use $x = a\sin\theta \Rightarrow \sqrt{a^2-x^2} = a\cos\theta$ (as above)
$dx = a\cos\theta\, d\theta$

Using a trig substitution, now you can compute it. Do so!

Hint: Along the way, you'll encounter the integral $\int \cos^2\theta\, d\theta$. The easiest way to compute this is to use the double angle identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ and then integrate. However, don't forget that you still have to express your answer back in terms of x.

$$a^2\int\cos^2\theta\,d\theta = a^2\int\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c =$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{\cancel{2}}\cdot\cancel{2}\sin\theta\cos\theta\right) + c =$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right) + c$$

$$= \boxed{\frac{1}{2}a^2\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2} + c}$$

1

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{\cancel{3\cos\theta}\, d\theta}{9\sin^2\theta \cdot \cancel{3\cos\theta}} = \frac{1}{9}\int \csc^2\theta \, d\theta =$$

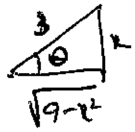
$x = 3\sin\theta$

$\sqrt{9-x^2} = \dots = 3\cos\theta$

$dx = 3\cos\theta \, d\theta$

$$= -\frac{1}{9}\cot\theta + c$$

$$\boxed{= -\frac{1}{9}\,\frac{\sqrt{9-x^2}}{x} + c}$$



$\Rightarrow \cot\theta = \dfrac{\sqrt{9-x^2}}{x}$

2

$$\int \frac{dx}{(4x^2-1)^{\frac{3}{2}}} = \int \frac{dx}{\left((2x)^2-1\right)^{\frac{3}{2}}} = \int \frac{\frac{1}{2}\sec\theta\tan\theta\, d\theta}{(\tan\theta)^3} =$$

$2x = \sec\theta$

$\left((2x)^2-1\right)^{\frac{3}{2}} = \left(\sec^2\theta - 1\right)^{\frac{3}{2}} = \left(\tan^2\theta\right)^{\frac{3}{2}} = (\tan\theta)^3$

$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta =$$

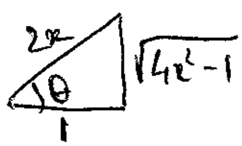
$$= \frac{1}{2}\int \frac{\cos^{\cancel{2}}\theta}{\cancel{\cos\theta}\cdot\sin^2\theta}\, d\theta =$$

$w = \sin\theta$

$dw = \cos\theta\, d\theta$

$$= \frac{1}{2}\int \frac{dw}{w^2} = -\frac{1}{2}w^{-1} + c = -\frac{1}{2w} + c = -\frac{1}{2\sin\theta} + c$$

$\sec\theta = \dfrac{2x}{1}$



$\Rightarrow \sin\theta = \dfrac{\sqrt{4x^2-1}}{2x}$

$$-\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{\sqrt{4x^2-1}} + c = \boxed{-\frac{x}{\sqrt{4x^2-1}} + c}$$

3

$$\int_3^4 \frac{1}{\sqrt{x^2-6x+10}}\, dx = \int_3^4 \frac{1}{\sqrt{(x-3)^2+1}}\, dx =$$

$x - 3 = \tan\theta$ $\quad x=3 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$; $\quad x=4 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\sqrt{(x-3)^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{\sec^{\cancel{2}}\theta\, d\theta}{\cancel{\sec\theta}} = \ln|\sec\theta + \tan\theta| \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} =$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln(\sqrt{2}+1) - \ln 1 = \boxed{\ln(\sqrt{2}+1)}$$