NAME: Solution Key Panther ID: ________________

Final Exam - MAD 2104 - Summer A 2014

1. (10 pts) Are $\neg(p \leftrightarrow q)$ and $p \leftrightarrow (\neg q)$ logically equivalent? Justify. Any style OK, but use at least one sentence in your justification.

Yes. Truth table is a good justification.
One other is to note that $\neg(p \leftrightarrow q)$ is True if and only if p, q have different truth values. The same for $p \leftrightarrow \neg q$.
Of course, this is equivalent with the truth table.

2. (16 pts) On the set of all integers, $\mathbf{Z}$, consider the congruency mod 5 relation, that is

$$m \equiv n \pmod 5 \text{ if and only if } 5 | (m - n).$$

(a) Show that the congruency mod 5 relation is an equivalence relation on $\mathbf{Z}$.

Reflexive: $\forall m \in \mathbb{Z}$ $m \equiv m \pmod 5$ since $5 | 0 = m - m$

Symmetric: If $m \equiv n \pmod 5 \Rightarrow 5 | (m-n) \Rightarrow m - n = 5k$ for some $k \in \mathbb{Z} \Rightarrow$

$\Rightarrow n - m = 5 \cdot (-k)$, with $(-k) \in \mathbb{Z}$ so $n \equiv m \pmod 5$

Transitive: If $m \equiv n \pmod 5$ and $(n \equiv q) \pmod 5 \Rightarrow$

$\Rightarrow m - n = 5k$ and $n - q = 5\ell$ for some $k, \ell \in \mathbb{Z} \Rightarrow$

$\Rightarrow m - q = (m-n) + (n-q) = 5(k+\ell)$, so $m \equiv q \pmod 5$

Since the relation is reflexive, symmetric & transitive, it is an equivalence relation

(b) Six integers are picked randomly. Show that among these six there are at least two integers whose difference is divisible by 5. For full credit here mention who are the pigeons, who are the pigeonholes (boxes), etc.

The pigeonholes (boxes) are the equivalence classes with respect to $\equiv \pmod 5$ relation also
$[0], [1], [2], [3], [4]$

Note $[0] = \{5k \mid k \in \mathbb{Z}\}$; $[1] = \{5\ell + 1 \mid \ell \in \mathbb{Z}\}$, etc.

Clearly $[0] \cup [1] \cup [2] \cup [3] \cup [4] = \mathbb{Z}$.

Since we pick 6 numbers (these are the pigeons) and there are 5 classes, there must be at least two numbers which fall in the same class. The difference of these two numbers must be

3. (10 pts) Write the sum below using summation notation and find the exact value of the sum.

$$4 + 9 + 14 + 19 + 24 + 29 + \ldots + 194 + 199 =$$

$$= \sum_{k=1}^{40} (5k-1) = 5\sum_{k=1}^{40} k - \sum_{k=1}^{40} 1$$

$$= 5 \cdot \frac{40 \cdot 41}{2} - 40 = \cancel{4100 - 40 = 7030}$$

$$= 4100 - 40 = 4060$$

4. (14 pts) Recursively define $a_0 = 1$, $a_1 = 3$, $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$.

(a) (2 pts) Calculate a_2, a_3.

$a_2 = 2a_1 + 3a_0 = 2 \cdot 3 + 3 \cdot 1 = 9$; $a_3 = 2a_2 + 3a_1 = 27$

(b) (2 pts) Guess a formula for a_n.

Guess $a_n = 3^n$ for all $n \geq 0$.

(c) (10 pts) Prove your guess using mathematical induction. Specify which type of induction you are using.

We use Strong Induction:

Basic steps: $n=0$ $\quad a_0 = 3^0 = 1$ ✓

$n=1$ $\quad a_1 = 3^1 = 3$ ✓

Assume $a_k = 3^k$ for all $0 \leq k \leq n$ (where $n \geq 2$)

Then $a_{n-1} = 3^{n-1}$ & $a_n = 3^n$, so

$$a_{n+1} = 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} = 2 \cdot 3^{n-1} + 3^{n-1} = (2+1) \cdot 3^{n-1} = 3^n.$$

So we proved that $P(n-1) \wedge P(n) \longrightarrow P(n+1)$

Since the basic steps $n=1, n=2$ are true, it follows that

$P(n)$ is true for all $n \geq 2$.

5. (24 pts) Answer and very briefly justify (4 pts each).

(a) How many edges does K_n have? Recall that K_n is the complete (simple, undirected) graph with n vertices.

$C(n,2) = \frac{n(n-1)}{2}$ Every two vertices are connected by an edge and there are $C(n,2)$ ways of choosing 2 vertices out of n.

(b) What is the coefficient of x^3y^5 in $(3x-2y)^8$?

By Binomial Theorem

$C(8,5)\cdot 3^3\cdot(-2)^5$ or $-\binom{8}{5}\cdot 3^3\cdot 2^5$ or $-\binom{8}{3}\cdot 3^3\cdot 2^5$

(c) What is the exact value of $P(10,3)$?

$P(10,3) = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$

(d) What is the value of the sum $C(n,0)+C(n,1)+C(n,2)+\ldots+C(n,n-1)+C(n,n)$?

2^n One of combinatorial identities
Both expressions count the number of subsets of a set with n elements

(e) A bowl contains 6 red balls, 9 yellow balls. What is the minimum number of balls which must be drawn (blindly) to be sure at least three balls of the same color are drawn?

5 Pigeonhole Principle $\lceil \frac{5}{2} \rceil = 3$

(f) Is there an undirected graph with three vertices of degree 3 and two vertices of degree 4? If yes, draw one such graph, if no, explain why it there is no such graph.

No. In any undirected graph there is an even number of vertices of odd degree.
This is a consequence of $\sum_{v\in V} \deg(v) = 2|E|$

6. (12 pts) Give your answer and a brief explanation for each of the following:

(a) How many bit strings of length nine contain exactly four 1s?

$C(9,4)$ ← the number of ways of choosing 4 positions out of 9.

(b) How many bit strings of length nine start with 11 or end with 00?

Principle of inclusion-exclusion

$2^7+2^7-2^5$

$|A\cup B| = |A|+|B|-|A\cap B|$

In our case: A = bit strings starting with 11

B = — " — ending with 00

(c) How many bit strings of length nine are palindromes?

The first five bits can each be chosen in two different ways. For the last four bits there is only one option in the case of palindromes.

2^5

7. (14 pts) Use mathematical induction to prove ONE of these.

(A) $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$, for any $n \geq 1$.

(B) Prove that for any positive integer n, a rectangular checkerboard of dimensions $(3 \cdot 2^n)$ by 2^n can be covered with right triominoes.

(A) Let $P(k)$ be the statement $1^2+3^2+5^2+\dots+(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$ for a given $k \geq 1$.

Basic Step: $P(1)$ $\quad 1^2 = \frac{1\cdot(2\cdot1-1)(2\cdot1+1)}{3}$ $\quad 1=1$ so $P(1)$ is true

Inductive Step: Assume $P(k)$ is true.

Want to show $P(k+1)$ is true.

$$\underbrace{1^2+3^2+5^2+\dots+(2k-1)^2}_{\text{Left side of } P(k)} + (2(k+1)-1)^2 = \frac{k\cdot(2k-1)(2k+1)}{3} + (2k+1)^2 =$$

inductive assumption

$$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right] = (2k+1)\cdot\frac{2k^2 \cancel{-k} + 6k+3}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3} = \frac{(2k+1)(k+1)(2k+3)}{3} = \frac{(k+1)(2(k+1)-1)(2(k+}{3}$$

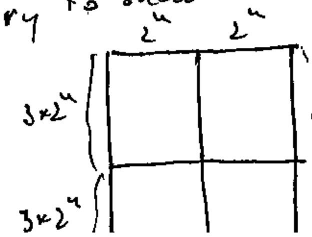
Thus ~~P(k+1)~~ we showed $P(k) \to P(k+1)$ so by induction $P(k)$ is true for all $k \geq 1$.

(B) Let $P(n)$ the statement: A $(3\times 2^n)\times 2^n$ board can be covered with troun

$P(1)$ $\quad (3\times2)\times 2 = 6\times2$ board can be covered as shown in the picture

Inductive step: Assume $P(n)$ true. Try to show $P(n+1)$ is true

$(3\cdot 2^{n+1})\times 2^{n+1}$ board can be split in 4 as shown



and use the inductive assumption for

boards.

8. (6 pts) Recall that CAE is a permutation of ACE. How many permutations of the letters $ABCDEFGH$ contain the string DAC? Explain your work.

Treat DAC as a single character. There are 5 more characters, so there are $P(6,6) = 6!$ permutations of ABCDEFGH that contain DAC

9. (10 pts) A club has 10 men and 15 women.

(a) In how many ways a committee of 2 men and 3 women can be selected?

$C(10,2) \cdot C(15,3)$ multiplicative rule

(b) How many ways are there to select a committee of 5 people and a president for this committee, if the president must be a woman.

Select the president first ← There are $C(15,1) = 15$ ways to do it
Then select the other 4 committee members out of remaining 14 peop
$C(14,4)$

So total # of ways $= 15 \times C(14,4)$

10. (10 pts) The English alphabet contains 21 consonants and 5 vowels. How many strings of six lowercase letters contain:

(a) at least one vowel?

$26^6 - 21^6$ complement rule

(b) exactly two vowels?

There are $C(6,2)$ ways of choosing the positions for the vowels.
So total number of strings with exactly two vowels is:
$C(6,2) \cdot 5^2 \cdot 21^4$ (repetitions of letters are allowed)

11. Give an algebraic proof (10pts) or a combinatorial proof (14pts) for the identity

$$kC(n,k) = nC(n-1,k-1), \text{ where } 1 \le k \le n .$$

You may give two proofs, but you will get credit for only one of the two (the higher score).

Combinatorial Proof: Count in how many ways a committee of k people and a president for this committee can be selected out of n people.

First way of counting: Choose the committee first ← $C(n,k)$ ways and then choose the president from the committee members (k ways to do

So total # is $k \cdot C(n,k)$ ways.

Second way of counting: Choose the president first out of all n people and then choose the other $(k-1)$ members of the committee out of $(n-1)$-people.

Total # is $n \cdot C(n-1,k-1)$ ways

Since the two methods count the same thing, we must have:

$$k\,C(n,k) = n \cdot C(n-1,k-1)$$

Algebraic Proof:

$$k \cdot C(n,k) = \cancel{k} \cdot \frac{n!}{(n-k)!\,\cancel{k!}_{(k-1)!}} = \frac{n!}{(n-k)!\,(k-1)!} = \frac{n \cdot (n-1)!}{(n-k)!\,(k-1)!} =$$

$$= n \cdot \frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!} = n \cdot C(n-1,k-1)$$

12. (12 pts) Describe all values of m, n for which the complete bipartite graph $K_{m,n}$ has
(a) an Euler circuit.
(b) an Euler path.
No need to find explicitly an Euler circuit or an Euler path, but explain clearly the result you are using.

From Euler's Theorem we know that an ~~[illegible]~~ undirected graph has an Euler circuit if and only if all vertices have even degree and has an Euler path if and only if all vertices have even degree except two which have odd degree.

In $K_{m,n}$, m vertices have degree n and n vertices have degree m.
So $K_{m,n}$ has an Euler circuit if and only if m, n are both even
$K_{m,n}$ has an Euler path if and only if (one of the m, n is 2 and the other is odd or $m = n = 1$).

13. (12 pts) What is the chromatic number of W_7? Justify your answer.

Solution slightly different than the one in class

Show that for a cycle C_n, $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$

Since W_n is obtained from C_n by adding one vertex and connecting it to <u>all</u> vertices of C_n
it is clear that the new vertex needs a new color

thus $\chi(W_n) = \chi(C_n) + 1 = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$

Thus $\chi(W_7) = 4$