Final Exam v.B - MAD 2014 - Summer A 2015 **NAME:** Solution Key

For full credit, when asked, you have to justify your answers. If "brief justification" is required, be brief, e.g "multiplicative rule" or "binomial theorem" will suffice (if correct). GOOD LUCK!

1. (15 pts)In each case, answer True or False. No justification is necessary for this problem – 3 pts each.

(a) For any sets A, B, $A \subseteq A \cap B$. False

(b) The set of finite length bit strings is countable. True

(c) The set $\{\sqrt{n} \mid n \in \mathbf{N}\}$ is countable. True

(d) If the adjacency matrix A of a graph is not symmetric, that is if $A^T \neq A$, then the graph is not undirected. True

(e) $P(n,k) = C(n,k) \cdot P(k,k)$, for all $0 \leq k \leq n$. True

2. (10 pts) Define a set S recursively by $5 \in S$ and $9 \in S$ and if $x, y \in S$ then $x + y - 2 \in S$. Is $11 \in S$? Justify.

Yes, $11 \in S$, because:

$$(5 \in S \wedge 5 \in S) \rightarrow 5+5-2 = 8 \in S$$

$$(5 \in S \wedge 8 \in S) \rightarrow 5+8-3 = \underline{\underline{11 \in S}}$$

3. (15 pts) Is it true that $(A \cup B) - C = (A - C) \cup (B - C)$ for all sets A, B, C? Answer and justify your answer. That is, prove the equality, or give a counter-example. A Venn diagram will help, but does not qualify as proof.

Yes, the statement is true:

distributive law

Proof: $(A \cup B) \setminus C = (A \cup B) \cap \overline{C} = (A \cap \overline{C}) \cup (B \cap \overline{C}) =$

(using set identities)

$$= (A \setminus C) \cup (B \setminus C)$$

Using logic is also fine.

4. (15 pts) Give your answer and a brief explanation for each of the following. You don't have to simplify answers.

(a) A saleswoman has to visit eight different cities. She must begin her trip in a specified city (you can assume it is Miami) and end her trip in another specified city of the eight (say, New York). She can visit the other six cities in any order she likes. How many possible orders can the saleswoman use when visiting these cities?

$P(6,6) = 6!$ permutations of the remaining 6 cities

(b) A club has 15 men and 11 women. In how many ways a committee of 3 men and 2 women can be selected? (The order of the people in the committee does not matter.)

$C(15,3) \cdot C(11,2)$ Multiplicative rule

(c) Another club has 8 men and 12 women. How many ways are there to select a committee of 4 people if the committee should contain at least one man and at least one woman? (The order of the people in the committee does not matter.)

Easiest way is probably to use the complement rule!

$$\binom{20}{4} - \binom{8}{4} - \binom{12}{4}$$

5. (15 pts) Use mathematical induction to prove ONE of these. Clearly indicate your choice.

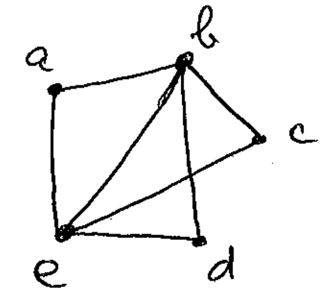
(A) Prove that 5 divides $n^5 - n$ for any non-negative integer n.

(B) $1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$, for any $n \geq 1$.

Both were part of the suggested exercises from the text, so I let you identify them in the text

6. (30 pts) Let G be a simple graph with vertices $\{a, b, c, d, e\}$, with the adjacency matrix A given below.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$



(a) (5 pts) Draw G (or ask me to do it, but at a cost of up to 5 points).

For each of the remaining parts answer and briefly justify (5 pts each).

(b) Does G have an Euler circuit? An Euler path?

G has an Euler circuit since all vertices have even degrees.
G does not have an Euler path

(c) Is G connected?

Yes G is connected, because any two vertices can be connected by a path. Note: You could try to prove that any graph that contains an Euler circuit or Euler path must be connected

(d) Is G bipartite?

No, G contains "triangles" so vertices cannot be colored just with two colors and fulfill the requirement that no edge connects two vertices of the same colo

(e) Is G isomorphic to W_4 (the Wheel with 5 vertices)?

No. The degree structure is different.
In W_4 one vertex has degree 4 and all the others have degree 3.
G contains vertices of deg. 2.

(f) What is $\chi(G)$, the chromatic number of G?

$\chi(G) = 3$

By explanation in (d) $\chi(G) \geq 3$.
But G can be colored with 3 colors:
e.g. b - blue, e - red, and (a, c, d) - green.

7. (20 pts) In each case answer and very briefly justify – 4 pts each.

(a) How many vertices does Q_5 have? Recall that Q_5 denotes the 5-cube graph.

2^5 - vertices in Q_5 are bit strings of length 5.

(b) What is the coefficient of y^4 in $(y+2)^{10}$? You don't have to simplify the answer.

$C(10,6)\cdot 2^6$ or $C(10,4)\cdot 2^6$

(c) What is the exact value of $P(6,3)$? What is the exact value of $C(10,1)$? (Writing the formula that leads to answer is enough justification.)

$P(6,3) = 6\cdot 5\cdot 4 = 120$

$C(10,1) = 10 = \frac{10!}{9!\cdot 1!}$

(d) What is the value of the sum $C(n,0) - C(n,1) + C(n,2) - C(n,3) + \ldots + (-1)^n C(n,n)$?

0, The above is the binomial expansion of $(1-1)^n$

(e) How many numbers must be selected from the set $\{1,2,3,4,5,6\}$ to guarantee that at least one pair of these numbers add up to 7?

4. Pigeonhole principle with holes being the subsets $\{1,6\}$, $\{2,5\}$, $\{3,4\}$.

8. Give an algebraic proof (10pts) or a combinatorial proof (15pts) for the identity

$$C(n,r)C(r,k) = C(n,k)C(n-k,r-k), \text{ where } 1 \le k \le r \le n\ .$$

You may give two proofs, but you will get credit for only one of the two (the higher score).

I leave you the algebraic proof.

Combinatorial proof: Count in two different way the following:

In how many ways can you choose from n people, a board of r people and then an executive committee for this board of only k people from the chosen r.

1st way of counting: choose the board first and then the executive committee from the board

$C(n,r)\cdot C(r,k)$ - total ways (from multiplicative property)

2nd way of counting: choose the executive committee first from all n people and then fill in the rest of the board from the remaining persons.

$C(n,k)\cdot C(n-k, r-k)$ - total ways

Since the two ways count the same thing

$C(n,r)\cdot C(r,k) = C(n,k)\cdot C(n-k, r-k)$

9. (15 pts) Give your answer and a brief explanation for each of the following. You don't have to simplify answers.

(a) How many bit strings of length 9 start with 000 or end with 11?

$2^6 + 2^7 - 2^4$ (inclusion-exclusion)

(b) How many bit strings of length 9 contain exactly four 0s?

$C(9,4)$ ← number of ways to choose the places where we put the four 0's.

(c) How many bit strings of length 9 contain exactly four 0s, with no two consecutive 0s? Hint: First place the 1s.

$\sqcup \; 1 \; \sqcup \; 1 \; \sqcup \; 1 \; \sqcup \; 1 \; \sqcup \; 1 \; \sqcup$

Box Box Box Box Box Box

We place the 1's. Then there are 6 boxes as shown in which we may or may not place one 0. There are $C(6,3)$ ways to choose the boxes with 0 so there are $C(6,3)$ strings as required.

10. (15 pts) Consider the function $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = \lfloor \frac{n}{5} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the number x (or "floor" of x).

(a) (4 pts) Find $f(10) =$ 2 $\qquad f(33) =$ 6

(b) (5 pts) Is the function f one-to-one? Justify your answer.

The function is not one-to-one

justification: $f(11) = f(10) = 2$

(c) (6 pts) Is the function f onto? Justify your answer.

Yes the function f is onto.

justification: Given $k \in \mathbb{N}$

$f(5k) = \lfloor \frac{5k}{5} \rfloor = k$