Name: Solution Key   Panther ID: ____________

Final Exam   **MAP 2302: Summer B 2019**

1. (14 pts) Answer True or False. No justification is necessary (unless the question looks ambiguous). (2 pts each)

(a) $y = 3e^{2x}$ is a solution for $y' = 2y$.   **True**   **False**

(b) The general solution for the equation $y'' - 9y = 0$ is $y = c_1e^{3x} + c_2e^{-3x}$, with c_1, c_2 constants.

True   **False**

(c) The UC method can be applied to find a particular solution of $y'' - 9y = x^2e^{2x}\sin x$.   **True**   **False**

(d) The function e^{x^2} has a Laplace transform.   **True**   **False**

(e) Given that $y = e^x$ is a solution of $(x^2 + x)y'' - (x^2 - 2)y' - (x + 2)y = 0$, a second linearly independent solution can be found using the substitution $y = e^xv$.

True   **False**

(f) The functions $\{\sin(2x), \cos(2x)\}$ are linearly dependent.   **True**   **False**

(g) The IVP problem $y' = xy^2$, $y(1) = 0$, has unique solution the trivial solution $y(x) = 0$.   **True**   **False**

2. (15 pts) Short answers:

(a) (3 pts) Suppose a simple harmonic motion (from Ch 5.2) has the formula $x(t) = 2\sin(t) + 3\cos(t)$. What is the amplitude of the motion?

$$A = \sqrt{2^2+3^2} = \sqrt{13}$$

(b) (3 pts) Give the standard form of a Bernoulli DE, from Ch. 2.3.B.

$$\frac{dy}{dx} + P(x)y = Q(x)\,y^n$$ (with $n \neq 0, n \neq 1$ as in those cases the DE is linear)

(c) (3 pts) Give the formula for an integrating factor μ, for a linear DE $y' + P(x)y = Q(x)$.

$$\mu = e^{\int P(x)\,dx}$$

(d) (6 pts) Find the singular points of the DE $(x^3 + x^2)y'' + y' + xy = 0$, and state whether they are regular singular or irregular singular points.

the DE can be rewritten as

$$y'' + \frac{1}{x^2(x+1)}y' + \frac{1}{x(x+1)}y = 0$$

so singular points are $x=0$ and $x=-1$

$x=0$ is an irregular singular point as $x \cdot \frac{1}{x^2(x+1)} = \frac{1}{x(x+1)}$ is NOT analytic at $x=0$

$x=-1$ is a regular singular point as $\cancel{(x+1)} \cdot \frac{1}{x^2\cancel{(x+1)}}$ and $(x+1)^2 \frac{1}{x\cancel{(x+1)}}$ are both analytic at $x=-1$.

3. (15 pts) Find the general solution (implicit form OK) for the first order DE

$$(x^2 + y^2)\,dx + (2xy + y^2)\,dy = 0\,.$$

The 1st order ΔE is homogeneous, but also exact

(since $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$).

It is much easier to solve it as an exact ΔE, although it is possible to solve it as homogeneous

Look for potential function F so that

$$\frac{\partial F}{\partial x} = x^2 + y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = (2xy + y^2)$$

Integrating the first (w.r.t. x)

$$F(x,y) = \int (x^2 + y^2)\,\partial x = \frac{1}{3}x^3 + y^2 x + g(y)$$

and plugging in the second

$$2xy + g'(y) = 2xy + y^2$$

so $g'(y) = y^2 \Rightarrow g(y) = \frac{1}{3}y^3 + c$

The implicit solution of the ΔE is thus $F(x,y) = c$

or $\boxed{\frac{1}{3}x^3 + y^2 x + \frac{1}{3}y^3 = c}$

4. (15 pts) Find the general solution for $y'' + 4y = 16e^{-2x}$. UC method is suggested, although other ways are possible (and acceptable too).

The homogeneous equation $y'' + 4y = 0$ has characteristic equation $\lambda^2 + 4 = 0$.

Thus $\lambda_{1,2} = \pm 2i$, so the complementary function is

$$y_c(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

We look for a particular solution $y_p(x) = Ae^{-2x}$

$$y_p'(x) = -2Ae^{-2x} \quad \text{and} \quad y_p''(x) = 4Ae^{-2x}$$

So, plugging in,

$$4Ae^{-2x} + 4Ae^{-2x} = 16e^{-2x}$$

$$8A = 16 \Rightarrow A = 2$$

Thus, $y_p(x) = 2e^{-2x}$ so the general solution of the given DE is

$$\boxed{y(x) = c_1 \cos(2x) + c_2 \sin(2x) + 2e^{-2x}}$$

5. (15 pts) Find a series solution in powers of x for the I.V.P. $y'' + xy' - 2y = 0$, $y(0) = 0$, $y'(0) = 1$. OK to list just the first three non-zero terms, but you should also list the recursive relation.

$x=0$ is an ordinary point, so we are guaranteed to have a solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ convergent in some interval $|x|<R$, with $R>0$.

From the initial conditions, we get

$$c_0 = y(0) = 0 \quad , 1\cdot c_1 = y'(0) = 1$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \quad , \quad y'(x) = \sum_{n=1}^{\infty} c_n \cdot n\, x^{n-1} \quad , \quad y''(x) = \sum_{n=2}^{\infty} c_n \cdot n(n-1) x^{n-2}$$

So the DE becomes.

$$\sum_{n=2}^{\infty} c_n \cdot n(n-1) x^{n-2} + \sum_{n=1}^{\infty} c_n \cdot n\, x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$\updownarrow k = n-2$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1) x^k + \sum_{k=1}^{\infty} c_k \cdot k\, x^k - 2\sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} + (k-2) c_k\right] x^k + 2\cdot 1\, c_2 - 2c_0 = 0$$

So we get $c_2 = c_0 = 0$ and

$$(k+2)(k+1) c_{k+2} + (k-2) c_k = 0 \quad \text{or} \quad c_{k+2} = -\frac{(k-2)}{(k+2)(k+1)} c_k \text{ for } k \geq 1$$

Using the recursive relation, we get

$$c_3 = -\frac{-1}{3\cdot 2} c_1 = \frac{1}{6} c_1 = \frac{1}{6} = \frac{1}{3!}$$

$$0 = c_0 = c_2 = c_4 = c_6 = \cdots \leftarrow c_{2k} = 0 \text{ for all } k$$

$$c_5 = -\frac{1}{5\cdot 4} c_3 = -\frac{1}{5!} \quad , \quad c_7 = -\frac{3}{7\cdot 6} c_5 = \frac{3}{7!}$$

So $y(x) = x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1\cdot 3}{7!}x^7 - \frac{1\cdot 3\cdot 5}{9!}x^9 + \ldots$

This can even be expressed with summation notation (exercise!)

6. (15 pts) Use a Laplace transform to solve this IVP, where δ is the usual Dirac delta function:

$$y' - 2y = \delta(t-3), \quad y(0) = 1 \,.$$

Simplify completely, writing the solution in piecewise form if necessary. Do not worry if $y(t)$ is not continuous.

Apply Laplace transformation to the given equation, ~~we~~ get

$$sY(s) - 1 - 2Y(s) = e^{-3s}$$

Thus $Y(s) = \frac{1}{s-2} + \frac{e^{-3s}}{s-2}$,

so $y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s-2}\right)$

or $y(t) = e^{2t} + u_3(t)\, e^{2\cdot(t-3)}$, where the second translation property was used for the second term.

In piece-wise form

$$y(t) = \begin{cases} e^{2t} & \text{for } 0<t<3 \\ e^{2t} + e^{2(t-3)} & \text{for } t>3 \end{cases}$$

equivalently, $y(t) = \begin{cases} e^{2t} & \text{for } 0<t<3 \\ e^{2t}\left(1+\frac{1}{e^6}\right) & \text{for } t>3 \end{cases}$

7. Choose ONE proof, but you could do TWO for possible bonus. Note the different values.

(A) (12 pts) Show that $L(e^{at}) = \frac{1}{s-a}$, for $s > a$.

(B) (18 pts) Compute the convolution of $\sin(bt)$ with itself and explain how the result is linked with the first formula 8 in the table prepared by Christian. Hint: you may need the identity $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

(C) (18 pts) Justify (as done in class or in the textbook) the formula $L\{\delta(t-t_0)\} = e^{-t_0 \cdot s}$

For (A) or (C) see class notes or textbook.

For (B), we compute $\sin(bt) * \sin(bt)$ using the definition of convolution

$$\sin(bt) * \sin(bt) = \int_0^t \sin(b\tau)\sin(b(t-\tau))\,d\tau$$

By the identity in the hint, the right side becomes

$$\frac{1}{2}\int_0^t \Big(\cos(b\tau - b(t-\tau)) - \cos(b\tau + b(t-\tau))\Big)\,d\tau =$$

$$= \frac{1}{2}\int_0^t \Big(\cos(2b\tau - bt) - \cos(bt)\Big)\,d\tau$$

$$= \frac{1}{2}\left[\frac{1}{2b}\sin(2b\tau - bt)\Big|_{\tau=0}^{\tau=t} - \tau\cos(bt)\Big|_{\tau=0}^{\tau=t}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2b}\sin(bt) - \frac{1}{2b}\sin(-bt) - t\cos(bt)\right]$$

$$= \frac{1}{2}\left[\frac{1}{b}\sin(bt) - t\cos(bt)\right]$$

Thus, $\sin(bt) * \sin(bt) = \frac{1}{2}\left[\frac{1}{b}\sin(bt) - t\cos(bt)\right]$.

Taking Laplace of both sides and using $L(f*g) = L(f)\cdot L(g)$

we get
$$\mathcal{L}\left(\frac{\sin(bt) - bt\cos(bt)}{2b}\right) = \mathcal{L}(\sin(bt))\cdot\mathcal{L}(\sin(bt))$$
$$= \frac{b^2}{(s^2+b^2)^2}$$

which is equivalent to formula (8) in the table from Christian.