Name: Solution Key Panther ID: ______________

Final Exam **MAP 2302: Summer B 2018**

1. (12 pts) Answer True or False. No justification is necessary (unless the question looks ambiguous). (2 pts each)

(a) The UC method can be applied to find a particular solution of $y'' + y = \sec x$ **True** **False**

$\sec x$ is not a UC function.

(b) Every solution of $y'' + 9y = 0$ can be expressed as $y(t) = c\cos(3t + \phi)$, for some constants c and ϕ. **True** **False**

See section 5.2 undamped free spring motion

(c) Given that $y = e^{2x}$ solves $y' = 2y$, a second linearly independent solution can be found by reducing the order.

True **False** $y' = 2y$ is a 1st order linear equation so all solutions are $y = c_1 e^{2x}$, so linear multiples of e^{2x}.

(d) Given that $y = e^x$ is a solution of $(x^2 + x)y'' - (x^2 - 2)y' - (x + 2)y = 0$, a second linearly independent solution can be found using the substitution $y = e^x v$.

True **False** reduction of order does apply here and $y = e^x \cdot v$ is exactly the substitution for reducing the order.

(e) The differential equation $(x^2 + x)y'' - (x^2 - 2)y' - (x + 2)y = 0$ has no singular points. **True** **False**

$x = 0$, $x = -1$ are singular points

(f) The IVP problem $y' = y^{1/3}$, $y(1) = 0$, has unique solution the trivial solution $y(x) = 0$. **True** **False**

see section 1.3

2. (~~15~~ 12 pts) For each differential equation below indicate its type (be specific) and write (only) the first step(s) towards solving it. For example, given the DE $y^{(4)} + 3y' + 2y = 0$, your answer should be: "linear homogeneous DE of order 4 with constant coefficients", and first step, "solve the characteristic equation $\lambda^4 + 3\lambda + 2 = 0$."
DO NOT spend time trying to solve completely any of the DEs in this problem. It is NOT required. (~~5~~ 4 pts each)

(a) $x^2y''-3xy'+2y = 0$

Cauchy-Euler DE
Substitution $x = e^t$

(b) $(x^2+y^2)\,dx-2xy\,dy = 0$

homogeneous 1st order DE
substitution $\frac{y}{x} = v$ (after writing it in the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$)

(c) $\frac{dx}{dt}+\frac{x}{t^2} = \frac{1}{t^2}$

Linear, 1st order DE

Multiply with integrating factor $\mu(t) = e^{\int \frac{1}{t^2}dt}$

Also acceptable: separable 1st order DE
separate variables $\frac{dx}{1-x} = \frac{dt}{t^2}$

or Bernoulli DE (non-linear) if rewritten as

$\frac{dt}{dx} = \frac{t^2}{1-x}$ and do the sub $v = t^{1-2}$

3. ($\cancel{15}$ 12 pts) Solve the I.V.P. $y' = -\frac{y}{x} + \frac{y^2}{x}$, $y(1) = 2$.

The DE is Bernoulli with $n=2$, so you could start with the substitution $v = y^{1-2} = y^{-1}$

and so on . .

But the DE is also separable

$$\frac{dy}{dx} = \frac{y^2-y}{x} \iff \frac{dy}{y^2-y} = \frac{dx}{x}$$

so $$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x}$$

$$\int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \int \frac{dx}{x}$$

by partial fractions

$$\frac{1}{y(y-1)} = \frac{y-(y-1)}{y(y-1)} = \frac{\cancel{y}}{\cancel{y}(y-1)} - \frac{\cancel{(y-1)}}{y\cancel{(y-1)}}$$

$$\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$$

$$\ln|y-1| - \ln|y| = \ln|x| + c$$

$$\ln\left|\frac{y-1}{y}\right| = \ln|x| + c$$

impose $y(1)=2$ to find c

$$\ln\left|\frac{2-1}{2}\right| = \ln 1 + c \Rightarrow c = \ln\left(\frac{1}{2}\right) = -\ln 2$$

Thus $\ln\left|\frac{y-1}{y}\right| = \ln|x| - \ln 2$

Because of the initial condition we can assume both $x>0$ and $y>1$ so drop the absolute values and exponentiate

$$\frac{y-1}{y} = \frac{x}{2} \Rightarrow 2y-2 = yx \Rightarrow -2 = y(x-2)$$

$$\Rightarrow \boxed{y = -\frac{2}{x-2} = \frac{2}{2-x}}$$

4. (~~15~~ 12 pts) Use Laplace transform to solve the system

$$\begin{cases} x' + y = e^{2t} \\ y' + x = 0 \end{cases}$$

with initial conditions $x(0) = 3,\ y(0) = 0$.

Let $\mathcal{L}(x(t)) = X$ and $\mathcal{L}(y(t)) = Y$ and apply $\mathcal{L}$ to the two equations.

$$\begin{cases} sX - x(0) + Y = \dfrac{1}{s-2} \\ sY - y(0) + X = 0 \end{cases} \iff \begin{cases} sX + Y = 3 + \dfrac{1}{s-2} \\ X + sY = 0 \end{cases}$$

Using Cramer's rule (or substitution, or elimination)

$$X(s) = \frac{\begin{vmatrix} 3+\frac{1}{s-2} & 1 \\ 0 & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ 1 & s \end{vmatrix}} = \frac{3s + \frac{s}{s-2}}{s^2-1} = \frac{3s}{(s-1)(s+1)} + \frac{s}{(s-2)(s-1)(s+1)} = \frac{3s^2-5s}{(s-2)(s-1)(s+1)}$$

$$Y(s) = \frac{\begin{vmatrix} s & 3+\frac{1}{s-2} \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} s & 1 \\ 1 & s \end{vmatrix}} = \frac{-\left(3 + \frac{1}{s-2}\right)}{s^2-1} = \frac{-3s+5}{(s-2)(s-1)(s+1)}$$

By partial fractions

$$X(s) = \frac{3s^2-5s}{(s-2)(s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1} \quad \text{with } A = \frac{2}{3},\ B = 1,\ C = \frac{4}{3}$$

$$Y(s) = \frac{-3s+5}{(s-2)(s-1)(s+1)} = \frac{\tilde{A}}{s-2} + \frac{\tilde{B}}{s-1} + \frac{\tilde{C}}{s+1} \quad \text{with } \tilde{A} = -\frac{1}{3},\ \tilde{B} = -1,\ \tilde{C} = \frac{4}{3}$$

Thus $x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\dfrac{\frac{2}{3}}{s-2} + \dfrac{1}{s-1} + \dfrac{\frac{4}{3}}{s+1}\right) = \dfrac{2}{3}e^{2t} + e^{t} + \dfrac{4}{3}e^{-t}$

$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\dfrac{-\frac{1}{3}}{s-2} - \dfrac{1}{s-1} + \dfrac{\frac{4}{3}}{s+1}\right) = -\dfrac{1}{3}e^{2t} - e^{t} + \dfrac{4}{3}e^{-t}$

5. (12 pts) Determine the value of the constant A such that $\underbrace{(Ax^2y + 2y^2)}_{M}\,dx + \underbrace{(x^3 + 4xy)}_{N}\,dy = 0$ is exact. Solve the resulting DE.

Test for exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \iff Ax^2 + 4y = 3x^2 + 4y$$

so $A = 3$ for the equation to be exact.

In this case, we look for $F(x,y)$ so that

$$\frac{\partial F}{\partial x} = 3x^2y + 2y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = x^3 + 4xy$$

$\Downarrow$

$$F(x,y) = \int (3x^2y + 2y^2)\,dx = x^3y + 2xy^2 + g(y)$$

Imposing the second condition, we get

$$x^3 + 4xy + g'(y) = x^3 + 4xy \quad \text{so} \quad g'(y) = 0 \quad \text{so}$$

$$g(y) = c \leftarrow \text{constant}$$

Thus $F(x,y) = x^3y + 2xy^2 + c$

So the solution (in implicit form) for the DE is

$$\boxed{x^3y + 2xy^2 = c}$$

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6. (~~15~~ pts) Find a series solution to the I.V.P. $y'' - 3xy = 0$, $y(0) = 1$, $y'(0) = 0$. OK to list just the first three non-zero terms, but you should also list the recursive relation.

$x=0$ is an ordinary point so we are guaranteed to find a convergent series solution $y = \sum_{n=0}^{\infty} c_n x^n$ on some interval around 0.

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} 3c_n x^{n+1} = 0$$

change the indices of summation

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} 3c_{n-1} x^n = 0$$

$$c_2 \cdot 2 \cdot 1 \cdot x^0 + \sum_{n=1}^{\infty} \left(c_{n+2}(n+2)(n+1) - 3c_{n-1} \right) x^n = 0$$

Thus we get $2c_2 = 0$ or $c_2 = 0$ and the recursive relation

$$c_{n+2} = \frac{3c_{n-1}}{(n+2)(n+1)} \quad \text{for } n \geq 1$$

From the initial conditions $y(0) = 1$, $y'(0) = 0$ we also get $c_0 = 1$ and $c_1 = 0$.

Thus we have $c_0 = 1$, $c_1 = 0$, $c_2 = 0$ and the recursive relation

$$c_{n+2} = \frac{3c_{n-1}}{(n+2)(n+1)} \quad \text{for } n \geq 1$$

We get $c_3 = \frac{3c_0}{3 \cdot 2} = \frac{1}{2}$, $c_4 = \frac{3c_1}{4 \cdot 3} = 0$, $c_5 = \frac{3c_2}{5 \cdot 4} = 0$

$c_6 = \frac{3c_3}{6 \cdot 5} = \frac{1}{20}$, $c_7 = c_8 = 0$

Thus $\boxed{y(x) = 1 + \frac{1}{2}x^3 + \frac{1}{20}x^6 + \dots}$

7. (12 pts) Find the Laplace transform $L(h(t))$, where

$$h(t) = \begin{cases} 2, & 0 < t < 3 \\ 0, & 3 < t < 6 \\ 2, & t > 6 \end{cases}$$

$$h(t) - 2u_0 = \begin{cases} 0 & , 0<t<3 \\ -2 & 3<t<6 \\ 0 & t>6 \end{cases}$$

$$h(t) - 2u_0 + 2u_3 = \begin{cases} 0 & 0<t<3 \\ 0 & 3<t<6 \\ 2 & t>6 \end{cases} = 2u_6$$

Thus $h(t) - 2u_0 + 2u_3 = 2u_6$ so

$h(t) = 2u_0 - 2u_3 + 2u_6$

$$L(h) = 2L(u_0) - 2L(u_3) + 2L(u_6)$$

$$L(h) = \frac{2}{s} - \frac{2e^{-3s}}{s} + \frac{2e^{-6s}}{s} = \frac{2(1-e^{-3s}+e^{-6s})}{s}$$

8. (12 pts + 6 pts bonus) If $F(s) = \frac{1}{s^2(s+3)}$, find $L^{-1}\{F(s)\}$ either using partial fractions or convolution. You'll get the bonus points if you solve the problem both ways.

with partial fractions

$$\frac{1}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$1 = As(s+3) + B(s+3) + Cs^2$$

$$A+C=0 \,,\quad 3A+B=0 \,,\quad 3B=1$$

$$B=\frac{1}{3} \,,\quad A=-\frac{1}{9} \,,\quad C=\frac{1}{9}$$

$$L^{-1}\left(\frac{1}{s^2(s+3)}\right) = L^{-1}\left(\frac{-\frac{1}{9}}{s} + \frac{\frac{1}{3}}{s^2} + \frac{\frac{1}{9}}{s+3}\right) = -\frac{1}{9} + \frac{1}{3}t + \frac{1}{9}e^{-3t}$$

with convolution

$$L^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{s+3}\right) = L^{-1}\left(\frac{1}{s^2}\right) * L^{-1}\left(\frac{1}{s+3}\right) = t * e^{-3t}$$

$$\text{But } t * e^{-3t} = \int_0^t x e^{-3(t-x)}\,dx = \int_0^t x \cdot e^{-3t} \cdot e^{3x}\,dx =$$

$$= e^{-3t}\int_0^t x \cdot e^{3x}\,dx = *$$

$$\int x e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$$

I.B.P.

$du = e^{3x}dx \qquad v = x$

$u = \frac{1}{3}e^{3x} \qquad dv = dx$

$$* = e^{-3t}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]\Bigg|_{x=0}^{x=t} = e^{-3t}\cdot e^{3t}\left[\frac{1}{3}t - \frac{1}{9}\right] - e^{-3t}\left(-\frac{1}{9}\right)$$

Thus $L^{-1}\left(\frac{1}{s^2(s+3)}\right) = \frac{1}{3}t - \frac{1}{9} + \frac{1}{9}e^{-3t}$ (same as with partial fractions)

9. Choose ONE proof, but you could do TWO for possible bonus. Note the different values.

(A) (12 pts) Show that $L(f') = sL(f) - f(0)$ (assume that f is nice and s is large enough).

(B) (12 pts) Show that $L(u_a(t)f(t-a)) = e^{-as}F(s)$, where $F(s) = L(f(t))$.

(C) (14 pts) Solve the I.V.P. $x'' + \lambda^2 x = \delta_{t_0}$, $x(0) = x_0, x'(0) = v_0$, where x_0, v_0, λ are known constants. Describe the behaviour of the solution $x(t)$ before and after the moment $t = t_0$. What practical situation could be modelled by the I.V.P. above? Hint: $L(\delta_{t_0}) = e^{-st_0}$.

Note: The values of ALL problems on this exam is changed to 12pts each. Exceptions are problem 8 where you could still get the bonus of 6 pts and problem 9 where you could receive 14 pts for option C.

For (A) or (B) consult the notes on the textbook

solution for (C): The IVP models the undamped motion of a spring on which at the moment t_0 a sudden exterior (Dirac delta) force is applied.

Apply Laplace transform on both sides and let $X = L(x)$.

$$s^2 X - sx(0) - x'(0) + \lambda^2 X = L(\delta_{t_0}) = e^{-st_0}$$

Thus $(s^2+\lambda^2)X = e^{-st_0} + sx_0 + v_0$ or

$$X(s) = \frac{e^{-st_0}}{s^2+\lambda^2} + \frac{sx_0}{s^2+\lambda^2} + \frac{v_0}{s^2+\lambda^2}$$

Thus $$x(t) = L^{-1}\left(\frac{sx_0}{s^2+\lambda^2}\right) + \frac{1}{\lambda}L^{-1}\left(\frac{v_0\cdot\lambda}{s^2+\lambda^2}\right) + \frac{1}{\lambda}L^{-1}\left(\frac{e^{-st_0}\cdot\lambda}{s^2+\lambda^2}\right)$$

$$x(t) = x_0\cos(\lambda t) + \frac{v_0}{\lambda}\sin(\lambda t) + \frac{1}{\lambda}u_{t_0}(t)\sin(\lambda(t-t_0))$$

or $$x(t) = \begin{cases} x_0\cos(\lambda t) + \frac{v_0}{\lambda}\sin(\lambda t) & \text{if } t<t_0 \\ x_0\cos(\lambda t) + \frac{v_0}{\lambda}\sin(\lambda t) + \frac{1}{\lambda}\sin(\lambda(t-t_0)) & \text{if } t>t_0 \end{cases}$$

Thus, up to the moment $t<t_0$, $x(t)$ has an oscillatory behavior (like the free, undamped motion of a spring), after the moment $t=t_0$ the perturbative term $\frac{1}{\lambda}\sin(\lambda(t-t_0))$ comes in.

Even for $t>t_0$ the motion will be oscillatory, but with a different amplitude