Name: ______________________________ **Panther ID:** _______________

Worksheet - Sep. 10 MAT 3501 Fall 2019

1. In each case, prove using induction:

(a) For all $n \geq 0$, $5 \cdot 2^{3n-2} + 3^{3n-1}$ is divisible by 19.

(b) (pb. 14 section 8.3) Show that the sum of the interior angles of a convex polygon is $180(n-2)$ degrees, where n is the number of sides of the polygon.

(c) For all $n \geq 1$, $1^3 + 2^3 + 3^3 + \ ... \ + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

(d) Show that $2^n|(n+1)(n+2) \ ... \ (2n)$, for all positive integers n.

(e) Assume that $y_1, y_2, ..., y_n$ are arbitrary positive numbers with $y_1 y_2 ... y_n = 1$. Prove that $y_1 + y_2 + ... + y_n \geq n$. When does equality hold?

2. Use problem 1 (e) to show that for any positive numbers $x_1, x_2, ..., x_n$,

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n}$$

This is the arithmetic mean vs geometric mean inequality. When does equality hold?

Definitions: (i) The **greatest common divisor** of a set of numbers is the largest of the divisors common to all the numbers. (*Notation:* gcd(...).)

(ii) The **least common multiple** of a set of numbers is the smallest of the multiples common to all the numbers. (*Notation:* lcm(...).) Note that this definition uses the Well Ordering Principle!

(iii) Two numbers a, b are relatively prime if they share no prime common factor, that is, if $\gcd(a,b) = 1$.

3. Use the Division Theorem and the Well Ordering Principle to prove the following important result:
Theorem: If a, b are positive integers, show that there exist integers x, y so that $ax + by = gcd(a, b)$.
Start of Proof: Consider the set $S = \{ax + by > 0 | x, y \in \mathbf{Z}\}$. By well ordering principle, the set S has a smallest element, call it d. Show that $d = gcd(a, b)$.

Corollary: If a, b are relatively prime, i.e. if $\gcd(a, b) = 1$, then there exist integers x, y so that $ax + by = 1$.