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Hermitian conformal classes and almost Kähler structures on 4-manifolds

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Abstract: It is shown that on most compact complex surfaces which admit symplectic forms, each Hermitian conformal class contains almost Kähler metrics. Results about the number of symplectic forms compatible to a given metric are obtained. As applications, alternative proofs for results of LeBrun on the Yamabe constants of Hermitian conformal classes are given, as well as some answers to a question of Blair about the isometries of almost Kähler metrics.

Keywords: Almost Kähler structures, Hermitian conformal classes, Yamabe constants, symplectic forms.

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1. Introduction

An *almost Hermitian structure* on an oriented manifold M^{2n} is a triple (g, J, ω) of a Riemannian metric g, an almost complex structure J, compatible with the orientation, and a non-degenerate 2-form ω, related by

$$\omega(X, Y) = g(X, JY),$$

for any tangent vectors $X, Y \in TM$. If the almost complex structure J is integrable, the triple (g, J, ω) is a *Hermitian structure*. If the form ω is closed, i.e., symplectic, then the triple (g, J, ω) is called an *almost Kähler structure*. Quite rarely, the two conditions, J integrable and ω closed, hold simultaneously, and in this case the triple (g, J, ω) defines a *Kähler structure* on the manifold. A metric will be called Kähler, Hermitian, or almost Kähler, if it admits a compatible corresponding structure. It is possible, and we show that this happens quite often, that a given metric is Hermitian and almost Kähler, but it is not a Kähler metric. One of our

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goals is to understand the relationship between the space of Hermitian metrics and the space of almost Kähler metrics on compact complex surfaces.

The subject of our paper could also fit into the wider context of the following problem: Given a closed, oriented manifold M, and a Riemannian metric g on M, determine if the metric g is Kähler, Hermitian, or almost Kähler, and also find how many structures of each type are compatible with the given metric. The answer to this problem is clear only in real dimension 2 where the notions of (almost) Kähler and Hermitian structures coincide. The dimension 4 is the next step to consider. It is well known that the holonomy group determines if a given metric is Kähler, in any dimension. In dimension 4, it is also understood fairly well when a given Riemannian metric admits (locally defined) compatible Hermitian structures, in the framework of so called *Riemannian Goldberg–Sachs theory* [3]. The number of such compatible structures is encoded in the structure of the self-dual part of the Weyl curvature [24, 3]. Riemannian metrics on 4-manifolds that admit a pair of distinct, compatible complex structures have been recently studied in [16, 23, 4]; it follows that "generically" on a *compact* oriented Riemannian 4-manifold, there is at most one globally defined positive orthogonal complex structure.

It seems more difficult to determine whether or not a given metric admits compatible almost Kähler structures, even in dimension 4. Recently, a strategy to do this has been outlined in [5] and this strategy has been used to show that certain Riemannian metrics cannot admit compatible almost Kähler structures (see also [22]). The number of compatible almost Kähler structures for a given metric is also not known. We treat these questions for the Riemannian metrics, compatible with the complex structure on a compact complex surface. As applications, we obtain some alternative proofs for results of LeBrun on the Yamabe constants of Hermitian conformal classes and give some answers to a question of Blair about the isometries of almost Kähler metrics.

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2. Statement of the main result

Let (M, g) be an oriented, Riemannian 4-manifold. The Hodge operator satisfies $*^2 = \mathrm{id}$ acting on the bundle of 2-forms and, therefore, we have the splitting

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

into self-dual 2-forms and anti-self-dual 2-forms, corresponding to the $+1$ and -1-eigenspaces of $*$. The well known correspondence between the (oriented) g-orthogonal almost complex structures and the self-dual forms for a Riemannian 4-manifold (M, g) imply that any self-dual, harmonic 2-form ω, of pointwise constant length $\sqrt{2}$, induces an almost Kähler structure (g, J, ω). Because of this equivalent definition, when the metric is fixed, we will very often just refer to the form when thinking at the almost Kähler structure.

Given a symplectic form ω on M, the *space of associated metrics to ω*, defined by

$$\mathbf{AM}_\omega = \{g \mid \text{(i)} \;\; \omega \in \Lambda^+ M, \;\; \text{(ii)} \;\; |\omega|_g = \sqrt{2}\},$$

is an infinite-dimensional, contractible space. Each metric from $\mathbf{AM}_\omega$ defines an almost Kähler structure with fundamental form ω. Because of condition (ii), there are no two elements in $\mathbf{AM}_\omega$

in the same conformal class of metrics. We define the *space of conformal associated metrics to* ω by

$$\mathbf{CAM}_\omega = \{g \mid \omega \in \Lambda^+ M\}.$$

Indeed, to justify the name, it is easily seen that

$$\mathbf{CAM}_\omega = \mathcal{C}^\infty_+(M) \cdot \mathbf{AM}_\omega,$$

where $\mathcal{C}^\infty_+(M)$ denotes the space of smooth, positive functions on M.

If $\mathbf{S}$ denotes the set of all symplectic forms on M, then the space of all almost Kähler metrics and the space of all conformal almost Kähler metrics are, respectively:

$$\mathbf{AK} = \bigcup_{\omega\in\mathbf{S}} \mathbf{AM}_\omega, \quad \mathbf{CAK} = \bigcup_{\omega\in\mathbf{S}} \mathbf{CAM}_\omega.$$

The following easy proposition motivates the questions we are addressing in this paper.

Proposition 1. *Let M be a closed, oriented* 4*-manifold, admitting symplectic structures.*

(a) *If ω and ω' are distinct, but cohomologous symplectic forms on M, then $\mathbf{CAM}_\omega \cap \mathbf{CAM}_{\omega'} = \emptyset$.*

(b) *Let g be a Riemannian metric on M. There exists a finite-dimensional vector subspace V of $\mathcal{C}^\infty(M)$, such that for any $f \in \mathcal{C}^\infty_+(M)$ with $f^2 \notin V$, the metric $g' = fg$ is not an almost Kähler metric.*

Proof. (a) Assume there exists a metric $g \in \mathbf{CAM}_\omega \cap \mathbf{CAM}_{\omega'}$. Let us recall that in dimension 4, harmonic 2-forms are invariant to conformal changes of metric, as also invariant is the splitting into self-dual and anti-self-dual forms. Then ω and ω' are both harmonic with respect to g. But by the Hodge decomposition theorem there is a unique harmonic representative in a given cohomology class. This contradicts the assumption $\omega \neq \omega'$.

(b) Given the metric g, assume that the metric $g' = fg$ is an almost Kähler metric. This is equivalent to the existence of a self-dual, harmonic 2-form ω' with

$$\frac{1}{f^2}|\omega'|^2_g = |\omega'|^2_{g'} = 2.$$

Let $\alpha_1, \ldots, \alpha_k$ form an orthogonal basis for the space of self-dual, harmonic 2-forms with respect to the global inner product induced by the metric g. Then, $\omega' = a_1\alpha_1 + \cdots + a_k\alpha_k$, for some constants $a_1, \ldots, a_k$. It follows that

$$2f^2 = \sum a_i a_j f_{ij},$$

where f_{ij} are the smooth functions given by the pointwise g-scalar product of α_i and α_j, $f_{ij} = (\alpha_i, \alpha_j)_g$. Taking V to be the space generated by the f_{ij}'s, the conclusion follows. □

A short way of rephrasing part (b) of Proposition 1 is that in a given conformal class most of the metrics are not almost Kähler. As for part (a), it leads to some questions. First, one may ask under what conditions two symplectic forms share a same associated metric. As we saw, this is not possible if the forms are cohomologous.

Question 1. *On a closed* 4*-manifold* M*, when can we find a Riemannian metric* g *admitting two different almost Kähler structures* (g, J_1, ω_1)*,* (g, J_2, ω_2) *with* $\omega_1 \neq \pm\omega_2$? *How many different almost Kähler structures could a given metric admit*?

It is well known that if M is a K3 surface or a torus and g is a hyper-Kähler metric, then g admits exactly an S^2-family of compatible almost Kähler (in fact, Kähler) structures. We would like to investigate other situations.

From a symplectic form ω, many others can be obtained by deforming the given one with "small" closed 2-forms. As symplectic forms in the same cohomology class have all disjoint sets of conformal associated metrics, it looks that many conformal classes contain almost Kähler metrics.

Question 2. *Find conformal classes which do not admit almost Kähler metrics.*

We consider Questions 1 and 2 for Hermitian metrics on compact complex surfaces. It makes sense to work only with compact, complex surfaces which also admit symplectic structures. Note that any closed complex surface with the first Betti number b_1 even admits Kähler structures, hence, in particular, symplectic structures. If b_1 is odd, the situation is more delicate and has been settled only recently (see [7, 14]).

Proposition 2. (O. Biquard [7]) *The only compact complex surfaces with* b_1 *odd which admit symplectic structures are primary Kodaira surfaces and blow-ups of these.*

Let (M, J) be a compact complex surface admitting symplectic forms. Denote by $\mathbf{H}$ the space of all Hermitian metrics compatible with the complex structure J. Here is our main result of this note.

Theorem 1. *Let* (M, J) *be a compact complex surface which admits symplectic structures.*

(a) *If* b_1 *is even then* $\mathbf{H} \subset \mathbf{CAK}$*. Moreover*:

- (a1) *Assume that* g *is a Kähler, non-hyper-Kähler metric on* M*, with Kähler form* ω*. Then* ω *and* $-\omega$ *are the only almost Kähler structures compatible to* g;
- (a2) *Assume that* g *is a non-Kähler, conformally-Kähler metric on* M*.*
  If $c_1 \neq 0$*, one of the following two situations occurs*: g *has exactly two* S^1 *families of associated almost Kähler structures, or* g *is not an almost Kähler metric.*
  If $c_1 = 0$*, one of the following three situations occurs*: g *has exactly two* S^1 *families of associated almost Kähler structures,* g *has exactly one* S^1 *family of associated almost Kähler structures, or* g *is not an almost Kähler metric.*

(b) *If* b_1 *is odd, there are two cases*:

- (b1) *If* (M, J) *is minimal then* $\mathbf{H} \subset \mathbf{CAK}$*. In this case, each metric* $g \in \mathbf{H} \cap \mathbf{AK}$ *has exactly one* S^1 *family of almost Kähler structures associated.*
- (b2) *If* (M, J) *is not minimal then* $\mathbf{H} \cap \mathbf{CAK} = \emptyset$*.*

The relation between the spaces $\mathbf{H}$ and $\mathbf{CAK}$ is a consequence of a result of P. Gauduchon ([13, Lemme II.3]; see also Proposition 3 in the next section). The main novelty of the theorem consists in the estimations on the number of almost Kähler structures compatible with a given metric. Regarding Question 1, we see that Kähler, non-hyper-Kähler metrics have an essentially unique compatible almost Kähler structure. However, as we see in (a2), there are examples of Hermitian, non-Kähler metrics, having S^1 families of compatible almost Kähler structures. These examples are strictly almost Kähler structures (i.e., non-Kähler), as it follows from (a1).

As an immediate consequence of (b2), we get an answer to Question 2.

Corollary 1. *On blow-ups of primary Kodaira surfaces, Hermitian conformal classes do not contain almost Kähler metrics.*

3. Proof of the main result

The proof relies on a series of propositions which we give below. Recall that the *Lee form* θ of an almost Hermitian 4-manifold (M, g, J) is defined by $dF = \theta \wedge F$, or equivalently $\theta = J\delta F$, where F denotes the Kähler form of (g, J), δ is the co-differential operator defined by g, and J acts by duality on 1-forms. (In this section and subsequently, we prefer to use F for the fundamental form of a non-Kähler, almost Hermitian or Hermitian structure, leaving ω to denote harmonic, self-dual forms.) It easily seen that $d\theta$ is a conformal invariant, that is, it depends on the conformal class of g and not on the metric itself. It is also known that Hermitian metrics with $d\theta = 0$ correspond to locally conformal Kähler metrics and the Hermitian metrics with $\theta = 0$ are, in fact, Kähler metrics.

A Hermitian metric such that the Lie form is co-closed, i.e., $\delta\theta = 0$, is called by Gauduchon a *standard Hermitian metric*. He proves in [11] the existence of standard metrics in each Hermitian conformal class (in any dimension) and its uniqueness modulo a homothety. In some sense, the standard Hermitian metric is the “closest” to a Kähler metric in its conformal class.

The first result we need is due to Gauduchon. For completeness, we give a proof, slightly different than the original argument in [13].

Proposition 3. (Gauduchon [13]) *On a compact complex surface M, endowed with a standard Hermitian metric g, the trace of a harmonic, self-dual form is a constant.*

Proof. Let (M, g, J, F) be the standard Hermitian structure on M. Any self-dual form $\alpha \in \Lambda^+M$ can be uniquely written as:

$$\alpha = aF + \beta + \bar{\beta}, \tag{1}$$

with $a \in \mathcal{C}^\infty(M)$ and $\beta \in \Lambda^{2,0}M$. We have to prove that if α is also (co)closed then a is a constant. Taking the divergence of both sides of (1), it follows

$$0 = J da + aJ\theta + \delta(\beta + \bar{\beta}).$$

Applying J to the above relation, we get:

$$da = -a\theta + J\delta(\beta + \bar{\beta}).$$

Taking inner product of both sides with da and integrating over the manifold implies

$$\int_M |da|^2 d\mu = -\int_M \frac{1}{2}(\theta, d(a^2))d\mu + \int_M (J\delta(\beta+\bar{\beta}), da)d\mu$$
$$= -\int_M \frac{1}{2}(\delta\theta, a^2)d\mu - \int_M ((\beta+\bar{\beta}), dJda)d\mu = 0,$$

since $\delta\theta = 0$ and $dJda \in \Lambda^{1,1}M$. Therefore $da = 0$, so a is a constant. □

Corollary 2. *Let (M, g, J, F) be a Hermitian surface. Then any harmonic, self-dual form ω is either the real part of a holomorphic $(2,0)$ form or is non-degenerate everywhere on M.*

This result already gives the relations between the spaces **H** and **CAK** stated in Theorem 1 at (a), (b1) and (b2). The next propositions deal with the number of compatible almost Kähler structures that various Hermitian metrics can have.

Lemma 1. *Let (M, J) be a complex manifold with $c_1 \neq 0$, equipped with a standard Hermitian metric g (which may be Kähler), and let F be the fundamental form. Suppose α_1, α_2 are two harmonic self-dual 2-forms which satisfy $\alpha_1^2 = \alpha_2^2$. Then the traces of these forms (which are necessarily constants) are equal up to sign:*

$$(\alpha_1, F) = \pm(\alpha_2, F).$$

Proof. By the known decomposition of 2-forms, α_1, α_2 can be written uniquely as:

$$\alpha_1 = a_1 F + \beta_1 + \bar{\beta}_1,$$
$$\alpha_2 = a_2 F + \beta_2 + \bar{\beta}_2,$$

where β_1, β_2 are $(2,0)$ forms and a_1, a_2 are constants. Now $\alpha_1^2 = \alpha_2^2$ is equivalent to

$$(a_1^2 - a_2^2)F^2 = 2(\beta_2 \wedge \bar{\beta}_2 - \beta_1 \wedge \bar{\beta}_1) = \mathrm{Re}\left((\beta_2 - \beta_1) \wedge (\bar{\beta}_2 + \bar{\beta}_1)\right).$$

By the assumption $c_1 \neq 0$, it follows that the form $\beta_2 - \beta_1$ must vanish at some point on M. From the above equality, as F^2 is a volume form on M and a_1, a_2 are constants, it follows $a_1^2 - a_2^2 = 0$. □

Proposition 4. *Let g be a standard Hermitian metric on a complex surface (M, J) with $c_1 \neq 0$, and let F be the fundamental form. Denote by ω the unique self-dual, harmonic form which has trace equal to 1 and is orthogonal to the space of holomorphic $(2,0)$ forms with respect to the cup product. Suppose α is a harmonic, self-dual form such that $\alpha^2 = \omega^2$ everywhere on M. Then $\alpha = \pm\omega$.*

Proof. By Lemma 1, $\mathrm{trace}\,\alpha = \pm\,\mathrm{trace}\,\omega = \pm 1$. Assume $\mathrm{trace}\,\alpha = \mathrm{trace}\,\omega = 1$. In this case α can be written as

$$\alpha = \omega + \mathrm{Re}(\beta),$$

where β is a holomorphic $(2,0)$ form. From $\alpha^2 = \omega^2$, it follows the relation

$$2\omega \wedge \mathrm{Re}(\beta) + \mathrm{Re}(\beta)^2 = 0,$$

everywhere on M. Integrating this relation on M, the first term vanishes because of the choice of ω. Therefore we get $\mathrm{Re}(\beta) = 0$, but as β is a $(2, 0)$ form this implies $\beta = 0$. Therefore we proved $\alpha = \omega$. Similarly, if $\mathrm{trace}\,\alpha = -\mathrm{trace}\,\omega = -1$, it follows that $\alpha = -\omega$. $\square$

Proposition 5. *Let g be a Kähler metric on M with Kähler form ω. Then either g is a hyper-Kähler metric, or $\pm\omega$ are the only almost Kähler structures compatible to g.*

Proof. If $c_1 \neq 0$ the conclusion follows immediately from the Proposition 4. Our argument below covers all cases.

Assume there exists another harmonic, self-dual form $\omega' \neq \pm\omega$, inducing same volume form as ω. Then ω' is uniquely written as

$$\omega' = a\omega + \eta,$$

where a is a constant and η is a smooth section of the canonical bundle. From ${\omega'}^2 = \omega^2$ we deduce

$$a^2 + \tfrac{1}{2}|\eta|^2 = 1,$$

hence $|a| \leqslant 1$. If $|a| = 1$, then $\eta = 0$, therefore $\omega' = \pm\omega$. If $|a| < 1$, we show that the metric g is in fact hyper-Kähler. Indeed, $\omega_1 = (1 - a^2)^{-\frac{1}{2}}\eta$ is a self-dual harmonic 2-form of length $\sqrt{2}$, pointwise orthogonal to ω, so it induces another almost Kähler structure on M, (g, J_1, ω_1), with J and J_1 anti-commuting. Since J is parallel with respect to the Levi-Civita connection of g, it follows that $(g, J_2 = J \circ J_1)$ is another almost Kähler structure, with J_2 anti-commuting with both J and J_1. Now, using an observation of Hitchin ([15], Lemma 6.8) that any triple of anti-commuting almost Kähler structures (g, J, J_1, J_2) defines a hyper-Kähler structure, we complete the proof. $\square$

Proposition 6. *Assume that g is a non-Kähler, conformally-Kähler metric on a compact complex surface (M, J).*

If $c_1 \neq 0$, one of the following two situations occurs: g has exactly two S^1 families of associated almost Kähler structures, or g is not an almost Kähler metric.

If $c_1 = 0$, one of the following three situations occurs: g has exactly two S^1 families of associated almost Kähler structures, g has exactly one S^1 family of associated almost Kähler structures, or g is not an almost Kähler metric.

Proof. First we will consider the case $c_1 \neq 0$. Let $g = fg'$, where $f \in \mathcal{C}^\infty_+$ and g' is a Kähler metric on (M, J) with Kähler form F. Let us assume also that (g, J, ω) is an almost Kähler structure. Then ω is a g-harmonic, self-dual form, of g-length $\sqrt{2}$ at every point on M. As g' is a conformal metric to g, the form ω is harmonic and self-dual with respect to g' as well. Hence there exists a constant $a \neq 0$ and a holomorphic $(2, 0)$ form β such that

$$\omega = aF + \mathrm{Re}(\beta).$$

But in this case, note that the forms

$$\omega_t^+ = aF + \cos(2\pi t)\,\mathrm{Re}(\beta) + \sin(2\pi t)\,\mathrm{Im}(\beta),$$

$$\omega_t^- = -aF + \cos(2\pi t)\,\mathrm{Re}(\beta) + \sin(2\pi t)\,\mathrm{Im}(\beta),$$

are also harmonic, self-dual and of length $\sqrt{2}$ with respect to the metric g, for any $t \in [0, 1]$. Therefore g has at least two S^1-families of almost Kähler structures compatible to g.

Suppose now that (g, J', ω') is some almost Kähler structure compatible to g and we would like to show that it must be one of the almost Kähler structures described by the the two S^1-families above. With the same reasoning as above

$$\omega' = a'F + \mathrm{Re}(\beta'),$$

where a' is a non-zero constant and β' is a holomorphic $(2, 0)$ form. Since $\omega^2 = \omega'^2$, by Lemma 1 we get $a' = \pm a$. Let us assume $a' = a$, the argument being similar in the other case. Now $\omega^2 = \omega'^2$ implies $\mathrm{Re}(\beta)^2 = \mathrm{Re}(\beta')^2$, which is equivalent to

$$\mathrm{Re}(\beta - \beta') \wedge \mathrm{Re}(\beta + \beta') = 0.$$

This means that at every point on M, the form $\mathrm{Re}(\beta + \beta')$ is collinear to $\mathrm{Im}(\beta - \beta')$. As both $\mathrm{Re}(\beta + \beta')$ and $\mathrm{Im}(\beta - \beta')$ are closed, we must have

$$\mathrm{Re}(\beta + \beta') = \lambda\, \mathrm{Im}(\beta - \beta'),$$

for λ a constant on M. The above relation implies

$$\beta' = \frac{\lambda^2 - 1}{\lambda^2 + 1}\beta - \frac{2\lambda}{\lambda^2 + 1} i\beta,$$

or, further,

$$\mathrm{Re}(\beta') = \frac{\lambda^2 - 1}{\lambda^2 + 1}\, \mathrm{Re}(\beta) + \frac{2\lambda}{\lambda^2 + 1}\, \mathrm{Im}(\beta).$$

It is easy to see now that ω' is in fact one of the forms in the family ω_t^+.

Next, let us consider the case $c_1 = 0$. By Kodaira's classification theorem we distinguish two sub-cases.

(i) (M, J) is a hyperelliptic surface or an Enriques surface. For these the dimension of the space of the harmonic self-dual 2-forms of any Riemannian metric is $b_+ = 1$, hence, by Proposition 1, there are no non-Kähler, globally conformal Kähler almost Kähler metrics.

(ii) (M, J) is a complex torus or a K3 surface. For these $b_+ = 3$ and they have hyper-Kähler metrics. Let us first remark that if g' is such a metric, then all self-dual, harmonic forms with respect to g' have constant length. Therefore there is no non-Kähler, almost Kähler metric which is conformal to a hyper-Kähler metric. However, a complex torus or a K3 surface do have Kähler metrics other than the hyper-Kähler ones. Choose one such metric and denote it again by g', the corresponding Kähler form being ω'. Suppose that $g = fg'$ is a non-Kähler, conformally Kähler metric which has an almost Kähler structure ω. Then we have

$$\omega = a\omega' + \mathrm{Re}(\beta),$$

where a is a real constant and β is a holomorphic $(2, 0)$ form. In fact, β is everywhere non-degenerate, so it is a holomorphic symplectic form on M.

Now we have two possibilities: if $a = 0$, then the metric g has one S^1 family of almost Kähler structures given by

$$\omega_t = \cos(2\pi t)\, \mathrm{Re}(\beta) + \sin(2\pi t)\, \mathrm{Im}(\beta);$$

if $a \neq 0$, then the metric g has two S^1 families of almost Kähler structures given by

$$\omega_t^{\pm} = \pm a\omega' + \cos(2\pi t)\,\mathrm{Re}(\beta) + \sin(2\pi t)\,\mathrm{Im}(\beta).$$

In either case, if g had other almost Kähler structures, it would follow that β has constant length with respect to g', which is a contradiction to the fact that g' is not hyper-Kähler. $\square$

We finally put together the above results to prove Theorem 1.

Proof of Theorem 1. Let us denote by p_g the geometric genus of the complex surface (M, J), i.e., the complex dimension of the space of holomorphic $(2, 0)$ forms. It is well known that $b_+ = 2p_g$ when b_1 is odd and $b_+ = 2p_g + 1$ when b_1 is even.

Let us consider first the case b_1 odd. By Proposition 2 of O. Biquard, the only compact complex surfaces that also admit symplectic structures are primary Kodaira surfaces (case of (b1)) and blow-ups of these (case of (b2)). For the primary Kodaira surfaces it is also known that they do admit holomorphic symplectic structures, that is, there exists a nowhere vanishing holomorphic $(2, 0)$ form. Denote such a form β and consider now a Hermitian metric g. The real form $\omega = \mathrm{Re}(\beta)$ is the real part of a holomorphic $(2, 0)$ form on M hence it is a harmonic, self-dual form for the metric g. As ω is also non-degenerate, there is a conformal metric to g such that ω and the new metric define an almost Kähler structure. We hence proved $\mathbf{H} \subset \mathbf{CAK}$ for primary Kodaira surfaces. Note also that if g is Hermitian, then any almost Kähler structure, say ω, has to be the real part of a holomorphic $(2, 0)$ form since $b_+ = 2p_g$. Hence $\omega = \mathrm{Re}(\beta)$, but then

$$\omega_t = \cos(2\pi t)\,\mathrm{Re}(\beta) + \sin(2\pi t)\,\mathrm{Im}(\beta)$$

is a whole S^1 family of almost Kähler structures compatible to the metric g. Finally, since for a primary Kodaira surface $b_+ = 2$, it follows from Proposition 1 that each Hermitian, almost Kähler metric has exactly one S^1 family of compatible almost Kähler forms.

To prove (b2) note first that if (M, J) is a blow-up of a primary Kodaira surface, then $c_1 \neq 0$ in this case. Let g be a Hermitian metric and let ω be a real, self-dual, harmonic form with respect to g. As above, since $b_+ = 2p_g$, it follows that $\omega = \mathrm{Re}(\beta)$, where β is a holomorphic $(2, 0)$ form. Since $c_1 \neq 0$, β must vanish at some point on M and so does ω. Therefore, for any Hermitian metric there are no harmonic, self-dual, everywhere non-degenerate forms.

Let us now consider the case b_1 even. In this case $b_+ = 2p_g + 1$, so for any Hermitian metric g, the space of real parts of holomorphic $(2, 0)$ forms is strictly contained in the space of all self-dual, harmonic forms. Let ω denote the (unique) self-dual, harmonic form which has trace equal to 1 and is orthogonal, with respect to the cup product, to the space of real parts of holomorphic $(2, 0)$ forms. This form is non-degenerate everywhere on M and hence for a conformal metric to g this form will define an almost Kähler structure. The statements from (a1) and (a2) follow from Propositions 5 and 6, respectively. $\square$

Remark 1. It would be nice to complete part (a) in Theorem 1 with a statement about the possible number of almost Kähler structures compatible to an arbitrary Hermitian metric (non-Kähler and not conformally Kähler). Proposition 4 shows that there are some Hermitian, non-Kähler metrics with a unique, up to sign, almost Kähler structure. However, we do not know a complete answer to this problem yet.

4. Yamabe and fundamental constants of Hermitian surfaces

The *Yamabe constant*, $Y(c)$, of the conformal class c on a compact 4-manifold M is defined to be

$$Y(c) = \inf_{g\in c} \left\{ \frac{\int_M s_g \, dV_g}{\sqrt{\int_M d\mu_g}} \right\},$$

where s_g is the scalar curvature of the Riemannian metric g and $d\mu_g$ denotes it volume form. It was proved by R. Shoen [25] that each conformal class c contains metrics of constant scalar curvature which realize the infimum in the above definition and, for this reason, these metrics are also referred as *Yamabe metrics*. We shall say that (M, c) is of *positive (resp. zero or negative) type* if $Y(c)$ is positive (resp. zero or negative).

It is a remarkable fact that the existence of metrics with positive scalar curvature on a compact 4-manifold leads to important information about the differentiable structure of the manifold. In particular, all Seiberg–Witten invariants must vanish. This was successfully used by C. LeBrun to prove that on a compact complex surface (M, J) with even first Betti number the existence of conformal classes (not necessarily compatible with J) of positive type forces (M, J) to have negative Kodaira dimension, i.e., to be either a rational surface, or a blow up of a ruled surface [17]. Considering only the conformal classes of Hermitian metrics, LeBrun's result was previously observed by several other authors [28, 27, 2]. The main idea dealing with Hermitian conformal classes is to use the Gauduchon's vanishing theorem, as it is explained below.

Let (M, J) be a compact complex surface and let c be a conformal class of Hermitian metrics on M. For any metric $g \in c$ we denote by u_g the *Hermitian scalar curvature* of (g, J), which is defined to be the trace of the Ricci form of the Chern connection ∇^c [13], i.e., we have

$$u_g = 2\langle R^c(F), F\rangle_g,$$

where R^c is the curvature of ∇^c and F, as usually, is the Kähler form of (g, J). Using the relation between the Chern connection ∇^c and the Riemannian connection ∇, given by (cf. [13, 27])

$$\nabla^c_X Y = \nabla_X Y - \tfrac{1}{2}\theta(Y)X - \tfrac{1}{2}\theta(JX)JY + \tfrac{1}{2}g(X, Y)\theta,$$

one can easily see (cf.[13]) that u_g and s_g are related by

$$u_g = s_g - \delta\theta + \tfrac{1}{2}|\theta|^2_g. \tag{2}$$

The eccentricity function $f_0(g)$ of a metric g in c is the positive function determined by the property $g = g_0/f_0(g)$, where g_0 is the standard metric of Gauduchon on c giving M a total volume 1 (different normalization than [6]). Note that a metric g is standard if and only if the corresponding function f_0 is a positive constant.

The *fundamental constant* $C(M, J, g)$ of a compact Hermitian surface we will define to be (compare with [6]):

$$C(M, J, g) = \int_M f_0(g) u_g \, d\mu_g.$$

Note that $C(M, J, g)$ does not depend on the choice of $g \in c$ and is a conformal invariant of c equal to $C(M, J, g_0) = \int_M u_{g_0}\, d\mu_{g_0}$, so we can denote it just as $C(M, J, c)$. It follows from (2) that $\int_M s_{g_0}\, d\mu_{g_0} \leqslant C(M, J, c)$ which gives the estimate

$$Y(c) \leqslant C(M, J, c), \tag{3}$$

with equality in (3) if and only if g_0 is a Yamabe–Kähler metric:

The fundamental constant $C(M, J, c)$ is closely related to the complex geometry of (M, J) in view of the following vanishing theorems of Gauduchon [12]:

Denote by P_m (resp.Q_m) the dimension of the space of holomorphic sections of $K^{\otimes m}$ (resp. of $K^{-\otimes m}$). Then we have:

(a) $C(M, J, c) > 0 \Longrightarrow P_m = 0, \forall m > 0$;
(b) $C(M, J, c) < 0 \Longrightarrow Q_m = 0, \forall m > 0$;
(c) $C(M, J, c) = 0 \Longrightarrow P_m = Q_m$ and $P_m \in 1, 0, \forall m > 0$.

In particular, for any positive conformal class c, the estimate (3) gives $C(M, J, c) > 0$, hence such a surface has to be of negative Kodaira dimension.

It is clear that except for the case when $P_m = Q_m = 0, \forall m > 0$ (some surfaces of negative Kodaira dimension), the sign of $C(M, J, c)$ is independent of c (see [6]). We also note that the existence of a Hermitian conformal class c with $C(M, J, c) = 0$ does imply the existence of a metric $g \in c$ of vanishing Hermitian scalar curvature u_g [6, Corollary 1.9], hence the Ricci form $R^c(F)$ (which represents up to multiplication with $1/2\pi$ the first real Chern class of (M, J)) is anti-self-dual. In particular, we have $c_1^2 \leqslant 0$ with equality if and only if $c_1 = 0$. So, on any complex surface (M, J) with Euler number χ and signature σ satisfying $2\chi + 3\sigma > 0$ (or $2\chi + 3\sigma = 0$ and $c_1 \neq 0$), the sign of $C(M, J, c)$ is also independent on the Hermitian conformal class c.

On the other hand, for a compact almost Kähler manifold (M, g, J, ω) we have another estimate for the Yamabe constant, coming from the basic inequality

$$\int_M s_g\, d\mu_g \leqslant 4\pi c_1 \cdot [\omega], \tag{4}$$

with equality if and only if the structure is Kähler. It follows from (4) that

$$Y(c) \leqslant 4\sqrt{2}\pi \frac{c_1 \cdot [\omega]}{\sqrt{[\omega] \cdot [\omega]}}, \tag{5}$$

with equality if and only if g is a Yamabe–Kähler metric.

To prove (4) one can consider the *first canonical connection* ∇^0, defined by Lichnerowicz in [20] to be

$$\nabla^0_X Y = \nabla_X Y - \tfrac{1}{2} J(\nabla_X J)(Y).$$

Since ∇^0 preserves J, its Ricci form γ^0 represents $2\pi c_1$, so using the above relation we obtain in the almost Kählerian case that $\langle \gamma^0, \omega \rangle = \frac{1}{2}s + \frac{1}{8}|\nabla J|^2$ (cf. [9]) which proves (4).

Now we shall use Theorem 1 to compare (3) and (5) on some Hermitian surfaces. We start with the following proposition, due to LeBrun in a more general setting [18]:

Proposition 7. *Let (M, g, J, F) be a Hermitian surface with b_1 even and let ω be a harmonic, self-dual form on M of non-negative trace. Then the following inequality holds*:

$$\int s\frac{|\omega|}{\sqrt{2}}\,d\mu \leqslant 4\pi c_1\cdot[\omega],$$

where s is the scalar curvature, $d\mu$ is the volume form and $|\cdot|$ is the pointwise norm determined by the metric g.

Proof. According to Corollary 2, we have two cases to consider.

Case 1: The form ω is non-degenerate everywhere on M. Denote by u the (strictly) positive function given by $\omega^2 = u^4F^2$, or, equivalently $\sqrt{2}u^2 = |\omega|$. The metric $g' = u^2 g$ is an associated metric for the symplectic form ω. The almost complex structure induced by g' and ω is homotopic to J, hence it has the same real first Chern class as J. Using (4), we get:

$$\int s_{g'}\,d\mu_{g'} \leqslant 4\pi c_1\cdot[\omega]. \tag{6}$$

Standard formulas for a conformal change of metric $g' = u^2g$ give

$$s_{g'} = u^{-2}s_g + 6u^{-3}\Delta_g u,$$
$$d\mu_{g'} = u^4\,d\mu_g.$$

From these we obtain

$$\int s_{g'}\,d\mu_{g'} = \int s_g\frac{|\omega|_g}{\sqrt{2}}\,d\mu_g + 6\int |du|_g^2\,d\mu_g \geqslant \int s_g\frac{|\omega|_g}{\sqrt{2}}\,d\mu_g, \tag{7}$$

and the proof is finished for the Case 1.

Case 2: The form ω is the real part of a holomorphic $(2,0)$ form. In this case we have $c_1\cdot[\omega] = 0$, since on a complex surface c_1 can be represented by a $(1,1)$ form (the Ricci form of a Hermitian connection). Consider ω_0 a harmonic, self-dual form, nowhere degenerate on M and denote

$$\omega_t = \omega_0 + t\omega,$$

for $t > 0$. Then ω_t are non-degenerate, harmonic self-dual forms for any t, so we can apply Case 1 to them. It follows

$$\int s\frac{|\omega_t|}{\sqrt{2}}\,d\mu \leqslant 4\pi c_1\cdot[\omega_t].$$

Taking into account that $c_1\cdot[\omega] = 0$, this becomes

$$\int s\frac{|\omega_0 + t\omega|}{\sqrt{2}}\,d\mu \leqslant 4\pi c_1\cdot[\omega_0],$$

and, after dividing by t,

$$\int \frac{s}{\sqrt{2}}\Big(\frac{|\omega_0|^2}{t^2} + \frac{2\langle\omega,\omega_0\rangle}{t} + |\omega|^2\Big)^{\frac{1}{2}}d\mu_g \leqslant \frac{4\pi}{t}c_1\cdot[\omega_0].$$

Taking the limit $t \longrightarrow \infty$, we obtain the conclusion in this case too. $\square$

Remark 2. A more careful application of relation (7) implies the inequality

$$\int s_g |\omega|_g \, d\mu_g + 6 \int \left| d(|\omega|^{1/2}) \right|_g^2 d\mu_g \leqslant 4\pi\sqrt{2} c_1 \cdot [\omega],$$

for any Hermitian metric g and any harmonic, self-dual form ω of non-negative trace. As a consequence, we see that on a scalar-flat Hermitian surface with b_1 even, all holomorphic $(2,0)$ forms have constant length.

Corollary 3. *Let (M, g, J, F) be a Hermitian surface with b_1 even and non-positive fundamental constant. The following inequality holds*:

$$\int s^2 \, d\mu \geqslant 32\pi^2 (c_1^+)^2,$$

where c_1^+ denotes the harmonic, self-dual part of c_1.

Proof. Apply Proposition 7 to the harmonic, self-dual form ω which satisfies $\omega = -c_1^+$. The fact that ω has non-negative trace holds because of the sign assumption on the fundamental constant. We get

$$4\sqrt{2}\pi (c_1^+)^2 \leqslant \int -s|\omega| \, d\mu \leqslant \int |s||\omega| \, d\mu.$$

Schwarz inequality implies

$$4\sqrt{2}\pi (c_1^+)^2 \leqslant \Big(\int s^2 \, d\mu \Big)^{\frac{1}{2}} \Big(\int |\omega|^2 \, d\mu \Big)^{\frac{1}{2}}.$$

Since ω is the harmonic representative of the class c_1^+, we have

$$\int |\omega|^2 d\mu = (c_1^+)^2,$$

and the conclusion follows. □

As already mentioned, on a rational surface (M, J) with $c_1^2 \geqslant 0$, the sign of $C(M, J, c)$ does not depend on the Hermitian conformal class c. Therefore it is always positive, since any rational surface admits a Kähler metric of positive total scalar curvature (cf. [28, 10]). With this observation and Proposition 7 in hand, we prove the following

Proposition 8. *Let (M, J) be a rational surface with $c_1^2 \geqslant 0$. Then for any Hermitian conformal class c on M we have*

$$Y(c) \leqslant 4\pi\sqrt{2(c_1^+)^2} \leqslant C(M, J, c), \tag{8}$$

where c_1^+ denotes the harmonic self-dual part of c_1. Moreover, equality in the right-hand side holds if and only if c contains a Kähler metric, while equality in the left-hand side holds if and only if c contains a Yamabe–Kähler metric.

Proof. Let $g \in c$ be an almost Kähler metric, with fundamental 2-form ω given by $\omega = F + \mathrm{Re}(\alpha)$, where F denotes the fundamental 2-form of the standard metric g_0 and α is a $(2, 0)$ form. The almost complex structure given by g and ω is homotopic to the complex structure J and hence they induce the same first Chern class, c_1. Denoting by $\gamma = R^c(F)$ the $(1, 1)$-Ricci form of (J, g_0), we have

$$\begin{aligned}\frac{c_1 \cdot [\omega]}{\sqrt{[\omega]\cdot[\omega]}} &= \frac{1}{2\pi}\frac{\int_M \gamma \wedge \omega}{\sqrt{\int_M \omega \wedge \omega}} = \\ &= \frac{1}{4\sqrt{2}\pi}\frac{\int_M u_{g_0}\, d\mu_{g_0}}{\sqrt{\int_M d\mu_{g_o} + \frac{1}{2}\int_M |\,\mathrm{Re}(\alpha)|^2\, d\mu_{g_0}}} \\ &\leqslant \frac{1}{4\sqrt{2}\pi} C(M, J, c)\end{aligned} \tag{9}$$

with equality if and only if $\mathrm{Re}(\alpha)$ vanishes, i.e., g_0 is a Kähler metric. On the other hand, since $b^+(M) = 1$ and $c_1 \cdot [\omega] > 0$ ([26,21]), we have that $(c_1)^+ = \lambda\omega$, for some positive real constant λ. Hence

$$\frac{c_1.[\omega]}{\sqrt{[\omega].[\omega]}} = \lambda\sqrt{[\omega].[\omega]} = \sqrt{(c_1^+)^2},$$

which after a substitution in (9) completes the proof of the right-hand side inequality of 8. The other inequality is a consequence of Proposition 7 and the above observation. □

Corollary 4. *Let (M, J) be as in Proposition* 8. *Then for any Hermitian conformal class c, the fundamental constant $C(M, J, c)$ satisfies*

$$C(M, J, c) \geqslant 4\pi\sqrt{2c_1^2}$$

with equality if and only if c contains a Kähler metric and the first Chern class has a self-dual representative with respect to c.

Corollary 5. *For any Hermitian conformal class c on $\mathbb{C}\mathbf{P}^2$ the Yamabe constant $Y(c)$ and the fundamental constant $C(M, J, c)$ satisfy*

$$Y(c) \leqslant 12\sqrt{2}\pi \leqslant C(M, J, g),$$

with equality in the right-hand side if and only if c contains a Kähler metric and with equality in the left-hand side if and only if c is conformally equivalent to the class of the Fubini-Study metric.

Proof. Since for $\mathbb{C}\mathbf{P}^2$ the negative second Betti number b_- vanishes we have that $4\pi\sqrt{2(c_1^+)^2} = 4\pi\sqrt{2c_1^2} = 12\sqrt{2}\pi$. The case of equality in the left hand side of the inequality follows from the observation that the only Kähler metric of constant scalar curvature on $\mathbb{C}\mathbf{P}^2$ is the Fubini–Study metric. □

Remark 3. The inequality $Y(g) \leqslant 12\sqrt{2}\pi$ was proved by LeBrun in [18] for an arbitrary conformal class on $\mathbb{C}\mathbf{P}^2$, investigating the "size" of the zero set of a self-dual form. As was noted there ([18, Corollary 3]), this estimate can be used to give an alternative proof of a result of Poon on the uniqueness of the self-dual structure of positive type on $\mathbb{C}\mathbf{P}^2$. Our Corollary 5, the fact that any Hermitian self-dual structure on $\mathbb{C}\mathbf{P}^2$ is of positive type (see [2]) and LeBrun's arguments give a simple proof in the framework of Hermitian geometry of the following:

Corollary 6. [2] *Any self-dual Hermitian conformal structure on $\mathbb{C}\mathbf{P}^2$ is equivalent to the standard one.*

5. Conformal transformations of almost Kähler metrics on 4-manifolds

D. Blair asked in [8] the following question: *given a compact almost Kähler manifold* (M^{2n}, g, J, ω) *and ϕ an isometry of the almost Kähler metric, is ϕ necessarily a symplectomorphism* (*or anti-symplectomorphism*)?

This is a particular case of our Question 1 and we use the results proven so far to give some answers in dimension 4. In fact, in our results ϕ will be a conformal transformation of the almost Kähler metric, i.e., the pull-back metric $\phi^* g$ is conformal to g. We first remark that Blair's question has an affirmative answer for compact 4-manifolds with $b_+ = 1$, as an easy consequence of Proposition 1. From the same Proposition 1, our next partial positive result also follows easily.

Proposition 9. *Let (M^4, g, J, ω) be a compact almost Kähler manifold and let ϕ be a conformal transformation of g, homotopic to the identity inside the group of diffeomorphisms of M. Then ϕ is an automorphism of the almost Kähler structure (g, J, ω).*

Proof. By assumptions, $\phi^*\omega$ is cohomologous to ω and $\phi^* g$ is conformal to g. Since $\phi^* g$ is an almost Kähler metric for the symplectic form $\phi^*\omega$, it follows that $g \in \mathbf{CAM}_\omega \cap \mathbf{CAM}_{\phi^*\omega}$. By Proposition 1 (a), this may hold only if $\phi^*\omega = \omega$, so ϕ is a symplectomorphism. To conclude that ϕ is also an isometry just note that a symplectic form cannot have two distinct, conformal associated metrics. □

Remark 4. Note that the above result is true in any dimensions if we assume ϕ to be an isometry in the identity component of the diffeomorphism group. It can be considered as a slight generalization of the well-known results of Lichnerowicz [19] about the connected group of isometries of a compact Kähler manifold.

The next result appears as a consequence of Theorem 1.

Theorem 2. *Let (M^4, g, J, ω) be a compact Kähler, non-hyper-Kähler surface. If ϕ is a conformal transformation of the Kähler metric then ϕ is a symplectomorphism or an anti-symplectomorphism.*

Proof. Let ϕ be a positive conformal isometry. Suppose that ϕ is not an isometry. Then $\phi^* g$ is an almost Kähler metric in the conformal class of g. Now, according to Theorem 1, (a2), we have that there is a whole S^1 family of almost Kähler structures with respect to the metric $\phi^* g$. Using ϕ^{-1}, we can induce a S^1-family of almost Kähler structures with respect to g, which contradicts with Theorem 1, (a1). So, ϕ must be an isometry. We use now Theorem 1, (a1) one more time to complete the proof. $\square$

Remark 5. The above result is closely related to [23, Theorem 5.3].

Now we will give examples when Blair's question has a negative answer. However, all such examples that we know so far are very special (all have $c_1 = 0$, for instance). It might be possible that in most instances isometries of almost Kähler metrics do indeed preserve (up to sign) the symplectic form.

Remark 6. The conclusion of Theorem 2 is no longer true for $T^4 = (S^1)^4$. Take the standard metric and consider the Kähler form $\omega = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4$. Let ϕ be the diffeomorphism which acts as identity on the first and third components and switches the second and the fourth. This is an isometry of the metric, but is clearly not an $\pm$-symplectomorphism. Hence Blair's question has a negative answer for T^4. Fore some special K3 surfaces such isometries (with respect to a hyper-Kähler metric) have been shown to exist by Alekseevsky and Graev [1]. Non-Kähler examples of this type can be given on T^4 (see [4]) and on primary Kodaira surfaces, which are T^2-bundles over T^2.

Remark 7. It may really happen that an isometry of an almost Kähler metric is an anti-symplectomorphism, as the following example shows:

Let $M^4 = S^2 \times S^2$ with the standard product metric. This metric is Kähler with respect to the form $\omega = \omega_1 - \omega_2$, the diffeomorphism taking one factor into the other is an isometry, but it is an anti-symplectomorphism of the form ω.

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