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The Kähler Cone Versus the Symplectic Cone

by
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Dedicated to the memory of a good friend, Vasile Ene

Abstract

Let $M^{2n},\ 2n \geq 4$, be a compact manifold admitting Kähler structures. A symplectic form ω on M that admits an associated Kähler structure will be called Kähler form. One suspects that, in general, the space of Kähler forms is quite "small" inside the space of symplectic forms. This is proved to be true, by giving examples o f elements in the difference of the two spaces. In dimension 4, cohomology classes that contain symplectic forms, but contain no Kähler forms representatives are found.

1 Introduction

Let M^{2n} be a compact, orientable manifold that admits symplectic structures. It is well known that a symplectic form ω on M determines:
(i) - a volume form $\sigma = \frac{1}{n!}\omega^n$, hence an orientation;
(ii) - non-trivial cohomology classes $[\omega^k] \in \mathbf{H}^{2k}(M, \mathbb{R})$;
(iii) - a homotopy class of almost complex structures.

In particular, we obtain Chern classes $c_i \in \mathbf{H}^{2i}(M, \mathbf{Z})$, for $i \in 1, ..., n$, and these are invariants of the symplectic structure (in fact, of the deformation class of the symplectic form). Taking various cup products

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and evaluating on the manifold, Chern numbers are defined and they all are symplectic invariants. Of particular importance for us will be $c_1 \cup [\omega^{n-1}](M)$ and $c_1^2 \cup [\omega^{n-2}](M)$ (here and throughout "$\cup$" denotes the cup product in cohomology).

Given a symplectic form ω on M, a Riemannian metric g is called associated to ω if there exists a g-orthogonal almost complex structure J such that

$$\omega(X, Y) = g(JX, Y), \tag{1.0.2.1}$$

for any tangent vectors $X, Y \in TM$. A triple (g, J, ω) of a Riemannian metric, an almost complex structure and a symplectic form satisfying the compatibility relation (1.0.2.1) is called an *almost Kähler structure* on M. A Riemannian metric that is part of such a structure will be called *almost Kähler metric*. There is an infinite dimensional (but contractible) space of almost Kähler metrics associated to a given symplectic form, and it would be useful to identify special metrics in this spac e. The most natural ones are *Kähler metrics*, i.e. associated metrics to ω for which the induced almost complex structure J is integrable. A symplectic form that admits an associated Kähler structure will be called *Kähler form*. It is now know n that the existence of a Kähler structure imposes strong restrictions on the topology of the manifold. Starting in 1976 with a work of Thurston [6], these topological restrictions have been used to provide many examples of compact symplectic manifold s that do not admit **any** Kähler forms. However, on a manifold M^{2n} that **does** admit Kähler structures, given a symplectic form ω, it is not clear, a priori, whether it is a Kähler form or not. The problem is easy for dimension $2n = 2$, where any almost complex structure is integrable, so any symplectic form is a Kähler form in this dimension. But such a state of affairs would seem to be very unlikely in dimensions $2n \geq 4$. (We will assume tacitly $2n \geq 4$ from now on.) Recently, a deep resul t of Taubes showed that any symplectic form on $\mathbf{CP}^2$ is equivalent to the standard Fubini-Study form, hence $\mathbf{CP}^2$ is a four-dimensional example where all symplectic forms are, in fact, Kähler forms. But $\mathbf{CP}^2$ is probably one of the rare exceptions. It is not hard to believe that in general the space of Kähler forms should be quite "small" inside the space of symplectic forms. Using the Chern numbers $c_1 \cup [\omega^{n-1}](M)$ and $c_1^2 \cup [\omega^{n-2}](M)$, we will be able to distinguish concrete examples of symplectic forms which are not Kähler forms. In fact, we will be able to say a bit more in dimension four.

Define the *cohomology Kähler cone*, $\mathcal{K}$, to be the set of cohomology classes in $\mathbf{H}^2(M, \mathbb{R})$ that have a Kähler form representative (note that we do not fix a complex structure on M). The word "cone" is appropriate because if $\alpha \in \mathcal{K}$ then $\lambda\alpha \in \mathcal{K}$, for any $\lambda > 0$. The *cohomology symplectic cone*, $\mathcal{S}$, is the set of cohomology classes in $\mathbf{H}^2(M, \mathbb{R})$ that have a symplectic form representative. In dimension four, we find classes in $\mathcal{S} \setminus \mathcal{K}$.

Theorem 1.1 *Let M^4 be a minimal complex surface of general type with $b^+ > 1$. Let (g, J, ω) be a Kähler structure on M and let β be a holomorphic (2,0) form on M with respect to the complex structure J. Denote by*

$$\alpha_t = (1-t)[\omega] + t[Re(\beta)] \ \in \mathbf{H}^2(M, \mathbb{R}).$$

Then we have the following:
(i) $\alpha_0 \in \mathcal{K}$;
(ii) $\exists t_0 \in (0,1)$ *such that for* $t \in (t_0, 1)$, $\alpha_t \in \mathcal{S} \setminus \mathcal{K}$;
(iii) $\alpha_1 \notin \mathcal{S}$.

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2 Main Results

Let (M^{2n}, g, J, ω) be an almost Hermitian manifold. The action of J extended to the bundle of 2-forms induces the following point-wise, orthogonal decomposition:

$$\Lambda^2 M = \mathbb{R}\omega \oplus \Lambda_0^{inv} M \oplus \Lambda^{anti} M,$$

where the factors denote respectively multiples of the fundamental form ω, J-invariant 2-forms of zero trace, and J-anti-invariant 2-forms. For a 2-form γ, the components with respect to this decomposition are:

$$\gamma = \frac{1}{n}(\gamma, \omega)\omega + \gamma_0^{inv} + \gamma^{anti}.$$

γ^{anti} denotes the J-anti-invariant part of γ. The J-invariant part is

$$\gamma^{inv} = \frac{1}{n}(\gamma, \omega)\omega + \gamma_0^{inv}.$$

If we complexify the tangent space and consider the usual decomposition of complex 2-forms in (2,0), (1,1) and (0,2) forms, it is easy to see that the

J-invariant forms are real parts of (1,1) forms and J-anti-invariant forms are real parts of (2,0) f orms (or, equivalently, of (0,2) forms).

By easy computation we have:

$$\gamma \wedge \omega^{n-1} = \frac{1}{n}(\gamma,\omega)\omega^n = (n-1)!\,(\gamma,\omega)\sigma; \tag{2.0.2.2}$$

$$\gamma \wedge \gamma \wedge \omega^{n-2} = (n-2)!\,\Big[\frac{n-1}{n}(\gamma,\omega)^2 - |\gamma_0^{inv}|^2 + |\gamma^{anti}|^2\Big]\sigma = \tag{2.0.2.3}$$
$$= (n-2)!\,[(\gamma,\omega)^2 - |\gamma^{inv}|^2 + |\gamma^{anti}|^2]\sigma.$$

Consider now $c \in \mathbf{H}^2(M,\mathbb{R})$ be a fixed, non-trivial cohomology class on a compact manifold M^{2n} and let the functional $\mathcal{F}_c : \mathbf{H}^2(M,\mathbb{R}) \longrightarrow \mathbb{R}$ be defined by

$$\mathcal{F}_c(\alpha) := (c \cup \alpha^{n-1}(M))^2 - (c^2 \cup \alpha^{n-2}(M))(\alpha^n(M)).$$

Assume that M admits a Kähler structure (g,J,ω). It is well known that for Kähler manifolds, the type decomposition of forms induced by J is preserved at the level of harmonic forms and hence for cohomology classes. The functional $\mathcal{F}_c$ is hel pful in distinguishing different types of cohomology classes on M, as the following Lemma 1 shows. The most important case of the Lemma for us, is the inequality

$$(c_1 \cup [\omega]^{n-1}(M))^2 \geq (c_1^2 \cup [\omega]^{n-2}(M))([\omega]^n(M)),$$

which holds for any Kähler form ω on M. This inequality is proved (in a slightly different way than we present below) by D. Perrone in [3], studying *cohomological Einstein Kähler manifolds.* These are Kähler manifolds satisfying the condition $c_1 = \lambda[\omega]$ and will also play an important role in our note.

Lemma 1 *Let (g,J,ω) be a Kähler structure on M^{2n} and consider $c \in \mathbf{H}^2(M,\mathbb{R})$.*
(a) If c is the real part of a non-trivial (2,0) class, then

$$\mathcal{F}_c([\omega]) < 0;$$

(b) If c is a non-trivial, real (1,1) class, then

$$\mathcal{F}_c([\omega]) \geq 0,$$

with equality if and only if $c = \lambda[\omega]$, for $\lambda \in \mathbb{R}$.

Proof: Part (a) follows immediately from relation (2.0.2.3) using the harmonic representative of c. To prove part (b), consider the bilinear form

$$b([\alpha],[\beta]) = [\alpha] \cup [\beta] \cup [\omega]^{n-2}(M),$$

defined on real (1,1) cohomology classes. From relation (2.0.2.3), we see that this is a symmetric form of Lorenz signature $(1, k-1)$, where $k = dim_{\mathbb{R}} \mathbf{H}^{1,1}$. Let γ be the harmonic representative of the class c. It decomposes further as

$$\gamma = a\omega + \xi,$$

where a is a constant (this uses again the fact that (g, J, ω) is a Kähler structure) given by $a = \frac{1}{n!} c_1 \cup [\omega]^{n-1}(M)$ and ξ is a trace free, harmonic, (1,1) form. By relation (2.0.2.3), we get $b([\xi],[\xi]) \leq 0$, hence $b(c - a[\omega], c - a[\omega]) \leq 0$ and this implies the inequality from the statement. Equality holds if and only if $c = a[\omega]$. $\square$

Let α_t be an infinitesimal deformation of $\alpha_0 = \alpha \in \mathbf{H}^2(M, \mathbb{R})$ with an initial direction $\alpha' = \frac{d}{dt}(\alpha_t)(0)$. Elementary calculations show that if $\alpha = \lambda c$, for $\lambda \in \mathbb{R}$, the n

$$\mathcal{F}_c(\alpha) = 0 \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{F}_c(\alpha_t)\Big)(0) = 0, \tag{2.0.2.4}$$

for any deformation α_t of α. Thus a cohomology class α such that $\alpha = \lambda c$ is a critical point for the functional $\mathcal{F}_c$. Moreover, at such a point, it can also be obtained easily

$$\frac{d^2}{dt^2}\Big(\mathcal{F}_c(\alpha_t)\Big)(0) = 2\frac{1}{\lambda^2}\mathcal{F}_{\alpha'}(\alpha). \tag{2.0.2.5}$$

Note that the second derivative of $\mathcal{F}_c(\alpha_t)$ at a point $\alpha = \lambda c$, depends only of the first jet of the deformation α_t and not on the second jet, as one might normally expect.

These computations and Lemma 1 imply immediately the following:

Proposition 2.1 *Assume that (g, J, ω) is a Kähler structure on M^{2n} and let $c = \lambda[\omega]$, for $\lambda \in \mathbb{R}$. Let also α_t be an infinitesimal deformation of $\alpha_0 := [\omega]$ with an initial direction $\alpha' = \frac{d}{dt}(\alpha_t)(0)$. Then*

$$\frac{d^2}{dt^2}\Big(\mathcal{F}_c(\alpha_t)\Big)(0) > 0, \ \textit{if } \alpha' \textit{ is a real, non-trivial (1,1) class;}$$

$$\frac{d^2}{dt^2}\Big(\mathcal{F}_c(\alpha_t)\Big)(0) < 0, \ \textit{if } \alpha' \textit{ is the real part of a non-trivial (2,0) class.}$$

We can now easily obtain examples of symplectic forms which are not Kähler forms.

Proposition 2.2 *Let (M^{2n}, g, J, ω) be a Kähler manifold with $c_1 = \lambda[\omega]$, for $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Assume that β is a (not identically zero) holomorphic (2,0) form on M and let ω_t be a deformation of ω suc h that*

$$\frac{d}{dt}(\omega_t)(0) = Re(\beta).$$

Then, for small values of $t \neq 0$, ω_t are symplectic forms, but they are not Kähler forms.

Proof: First of all, ω_t are symplectic forms for small values of t, since the non-degeneracy is an open condition. To show that they are not Kähler forms, note that from Proposition 2.1, we get

$$\frac{d^2}{dt^2}\Big(\mathcal{F}_{c_1}([\omega_t])\Big)(0) < 0.$$

Since (2.0.2.4) holds, it follows that $\mathcal{F}_{c_1}([\omega_t]) < 0$ for small values of $t \neq 0$. The result is then a consequence of Lemma 1. □

Remark 1: Note that, although it is very tempting, we cannot conclude that the cohomology classes $\alpha_t := [\omega_t]$ are in $\mathcal{S}$, but not in $\mathcal{K}$. The problem is that it is not known in general if cohomologus symplectic forms have necessarily the same Chern class c_1.

Remark 2: A small, linear deformation of any Kähler class in the direction of a (1,1) class will still be Kähler, with respect to the same complex structure J. It would be interesting to know if the result of the Proposition 2.2 is true for all Kähler forms, not necessarily satisfying $c_1 = \lambda[\omega]$.

Consider now a linear deformation of a Kähler form in the direction of the real part of a holomorphic (2,0) form, on a 4-dimensional Kähler manifold.

Proposition 2.3 *Let (M^4, J) be a complex surface of general type with $c_1^2(M) > 0$ and $b^+ > 1$. Let (g, J, ω) be a Kähler structure and β a non-trivial holomorphic (2,0) form on M. Denote $\omega_t = \omega + tRe(\beta)$, for $t \in \mathbb{R}$. Then ω_t is a symplectic form for any value of $t \in \mathbb{R}$, but for $|t|$ sufficiently large, ω_t cannot be a Kähler form. Moreover, if the the Kähler surface satisfies $c_1 = \lambda[\omega]$, for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, then ω_t cannot be a Kähler form for any value $t \neq 0$.*

Proof: Consider again the function

$$\mathcal{F}_{c_1}([\omega_t]) := (c_1 \cup [\omega_t](M))^2 - (c_1^2(M))([\omega_t]^2(M)).$$

Note that

$$\omega_t^2 = (\omega + tRe(\beta))^2 = \omega^2 + t^2 Re(\beta)^2 = (1 + \frac{t^2}{2}|Re(\beta)|^2)\omega^2,$$

thus ω_t is non-degenerate for any value of t, hence symplectic. We directly compute:

$$\mathcal{F}_{c_1}([\omega_t]) = (c_1 \cup [\omega](M))^2 - (c_1^2(M))([\omega]^2(M)) - \frac{(c_1^2(M))([Re(\beta)]^2(M))}{2}t^2,$$

so for large values of $|t|$, $\mathcal{F}_{c_1}([\omega_t]) < 0$. If $c_1 = \lambda[\omega]$, $\mathcal{F}_{c_1}([\omega_t]) < 0$ for any $t \neq 0$, so the result follows from Lemma 1. □

Remark 3: It can be easily seen that the metric $\tilde{g}_t = (1 + \frac{t^2}{2}|Re(\beta)|^2)^{1/2} g$ is an associated almost Kähler metric for the symplectic form $\omega_t = \omega + tRe(\beta)$. This observation is elaborated in [1], where we study almost Kähler metrics conformally equivalent to Hermitian metrics on compact complex surfaces.

We now prove Theorem 1.1 stated in the introduction. The condition $b^+ > 1$ in Theorem 1.1 and Proposition 2.3 assures the existence of non-trivial holomorphic (2,0) forms on M. Indeed, for a complex surface of Kähler type, $b^+ = 2p_g + 1$, where p_g is the geometric genus of the complex surface, i.e. the complex dimension of the space of holomorphic (2,0) forms. Note also that a minimal complex surface of general type admits Kähler structures and satisfies $c_1^2(M) > 0$.

Proof of Theorem 1.1: Part (i) of the theorem is trivial. We will prove (ii) and (iii) using some simple facts about Seiberg-Witten invariants. Denote by $K := \Lambda^{2,0}$ the (isomorphism class of the) canonical bundle induced by (J, ω) on M. It is known (see, for example, [2]) that on a minimal complex surface of general type the only Seiberg-Witten basic classes (i.e. classes that induce non-zero invariants) are $\pm K$. For a different symplectic form $\tilde{\omega}$, let $\tilde{K}$ be the canonical bundle induced by $\tilde{\omega}$. By [4], $\tilde{K}$ is a Seiberg-Witten basic class, so $\tilde{K} = \pm K$. The sign ambiguity is resolved by another result of Taubes [5] stating that

$$c_1(\tilde{K}) \cup [\tilde{\omega}](M) \leq 0,$$

with equality if and only if $c_1(\tilde{K}) = 0$. So, up to sign, all symplectic forms on a minimal complex surface of general type induce the same first Chern class. By a direct computation

$$\mathcal{F}_{c_1}(\alpha_t) = (1-t)^2\mathcal{F}_{c_1}(\alpha) - t^2(c_1^2(M))((Re\beta)^2(M)),$$

and now part (ii) of the Theorem follows from Lemma 1.
For part (iii), assume there exists a symplectic form $\tilde{\omega}$ in the cohomology class $\alpha_1 = [Re(\beta)]$. Since $[\beta]$ is a (2,0) class and $c_1(\tilde{K}) = \pm c_1(K)$ is a (1,1) class with respect to J, it follows that $c_1(\tilde{K}) \cup [\tilde{\omega}](M) = 0$. Thus $c_1(\tilde{K}) = c_1(K) = 0$, but this contradicts the assumption of M being complex surface of general type. $\square$

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