The Greek Letters

Chapter 17

Example (Page 365)

- A bank has sold for \$300,000 a European call option on 100,000 shares of a non-dividend-paying stock (the price of the option per share, "textbook style", is therefore \$3.00).

- S_0 = 49, K = 50, r = 5%, σ = 20%, T = 20 weeks, μ = 13%

- (Note that even though the price of the option does not depend on μ, we list it here because it can impact the effectiveness of the hedge)

- The Black-Scholes-Merton value of the option is \$240,000 (or \$2.40 for an option on one share of the underlying stock).

- How does the bank hedge its risk?

Naked & Covered Positions

- Naked position:
  - One strategy is to simply do nothing: taking no action and remaining exposed to the option risk.
  - In this example, since the firm wrote/sold the call option, it hopes that the stock remains below the strike price ($50) so that the option doesn't get exercised by the other party and the firm keeps the premium (price of the call it received originally) in full.
  - But if the stock rises to, say, $60, the firm loses (60-50)x100,000: a loss of $1,000,000 is much more than the $300,000 received before.

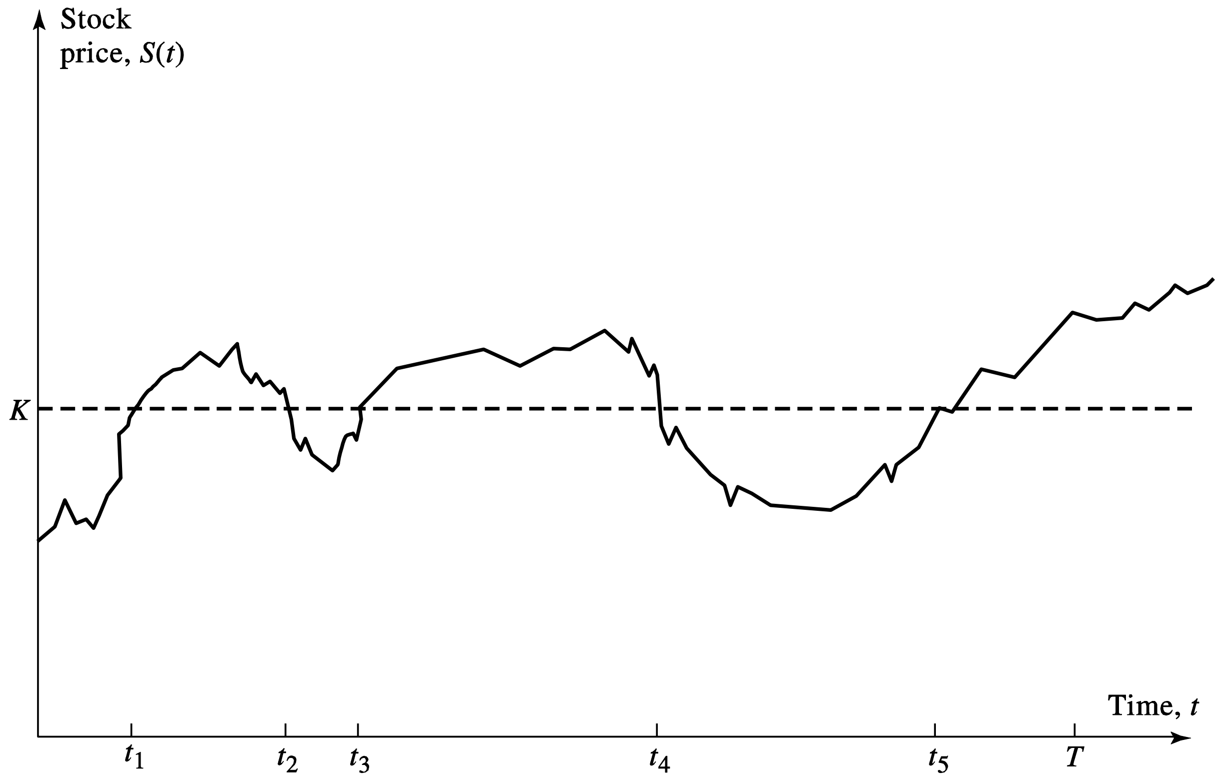
- Covered position:
  - The firm can instead buy 100,000 shares today in anticipation of having to deliver them in the future if the stock rises.
  - If the stock declines, however, the option is not exercised but the stock position is hurt: if the stock goes down to $40, the loss is (49-40)x100,000 = $900,000.

Stop-Loss Strategy

- The stop-loss strategy is designed to ensure that the firm owns the stock if the option closes in-the-money and does not own the stock if the option closes out-of-the-money.

- The stop-loss strategy involves:
  - Buying 100,000 shares as soon as the stock price reaches $50.
  - Selling 100,000 shares as soon as the stock price falls below $50.

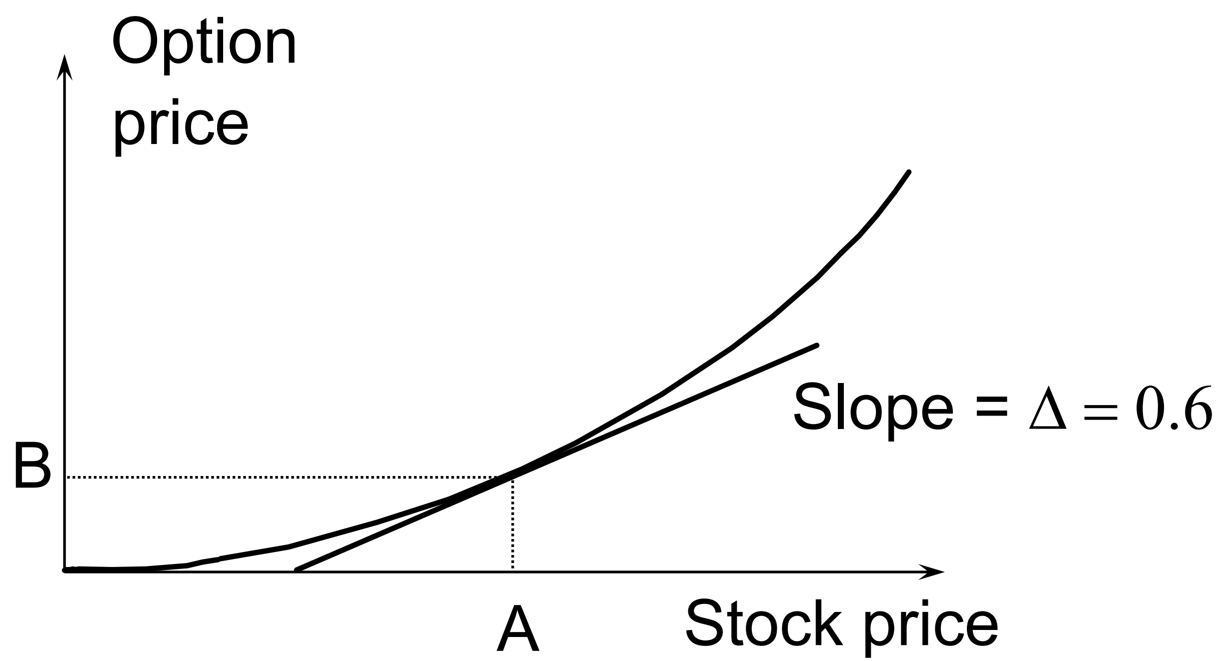
Stop-Loss Strategy continued



However, repeated transactions (costly) and the fact that you will always buy at K+ε and sell at K-ε makes this strategy not a very viable one.

Delta (See Figure 17.2, page 369)

- Delta (Δ) is the rate of change of the option price with respect to the underlying



Hedge

- Trader would be hedged with the position:
  - Short/wrote 1000 options (each on one share)
  - buy 600 shares

- Gain/loss on the option position is offset by loss/gain on stock position

- Delta changes as stock price changes and time passes.

- Hedge position must therefore be rebalanced often, an example of *dynamic hedging*.

Delta Hedging

- Delta hedging involves maintaining a delta neutral portfolio: a position with a delta of zero.

- The delta of a European call on a non-dividend-paying stock is $N(d_1)$

- The delta of a European put on the stock is $[N(d_1) - 1]$ or equivalently: $-N(-d_1)$.

First Scenario for the Example:
Table 17.2 page 372

Week	Stock price	Delta	Shares purchased	Cost (‘$000)	Cumulative Cost ($000)	Interest ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
…….	…….	…….	…….	…….	…….	…….
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0	5263.3	

First Scenario results

- At the end, the option is in-the-money.
- The buyer of the call option exercises it and pays the strike of 50 per share on the 100,000 shares held by the firm.
- This revenue of $5,000,000 partially offsets the cumulative cost of $5,263,300 and leaves a net cost of $263,300.

Second Scenario for the Example

Table 17.3 page 373

Week	Stock price	Delta	Shares purchased	Cost ('$000)	Cumulative Cost ($000)	Interest ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
…….	…….	…….	…….	…….	…….	…….
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

Second Scenario results

- At the end, the option is out-of-the-money.
- The buyer of the call option does not exercise it and nothing happens on that front.
- The delta having gone to zero, the firm has no shares left.
- The net cumulative cost of hedging is $256,600.

Comments on dynamic hedging

- Since by hedging the short/written call the firm was essentially *synthetically replicating* a long position in the option, the (PV of the) cost of the replication should be close to the Black-Scholes price of the call: $240,000.

- They differ a little in reality because the hedge was rebalanced once a week only, instead of continuously.

- Also, in reality, volatility may not be constant, and there are some transaction costs.

Delta of a Portfolio

- The concept of delta is not limited to one security.
- The delta of a portfolio of options or derivatives dependent on a single asset with price S is given by $\Delta_\Pi = \Delta\Pi/\Delta S$.
- If a portfolio of *n* options consists of a <u>quantity</u> q_i of option *i*, the delta of the portfolio is given by:

$$\Delta_\Pi = \sum_{i=1}^{n} q_i \Delta_i$$

Delta of a Portfolio: example

- Suppose a bank has the following 3 option positions:
  - A long position in 100,000 call options with a delta of 0.533 for each.
  - A short position in 200,000 call options with a delta of 0.468 for each.
  - A short position in 50,000 put options with a delta of -0.508 for each.

- The delta of the whole portfolio is:
  - Δ_{Π} = 100,000x0.533 – 200,000x0.468 – 50,000x(-0.508)
  - Δ_{Π} = -14,900

- The portfolio can thus be made delta neutral by buying 14,900 shares of the underlying stock.
- It also means that when the stock goes up by $1, the portfolio goes *down* by $14,900.

Theta

- The Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of its value with respect to the passage of time.

- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long call or put option declines.

- It is sometimes referred to as the *time decay* of the option.

Theta

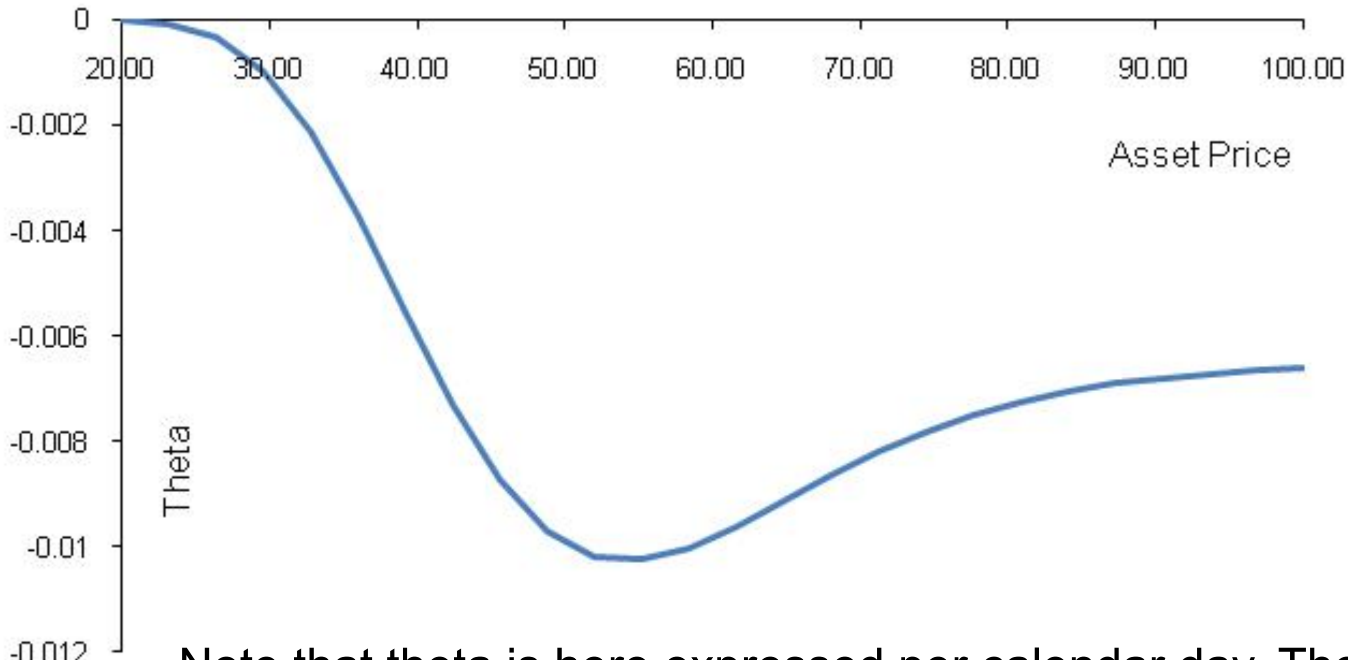
- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

$$\Theta(call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

$$with \ \ N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\Theta(put) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Theta for Call Option: K=50, σ = 25%, r = 5%, T = 1



Note that theta is here expressed per calendar day. The theta obtained from the formula is annual. For example, for S=50, θ=-3.6252 and so we have θ = -3.6252/365 = -0.01 per calendar day.

Gamma

- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset, so it is the “change of the change”.
- Gamma is greatest for options that are close to being at-the-money since this is where the slope, delta, changes the most.
- If Gamma is high, then delta changes rapidly.
- If Gamma is low, then delta changes slowly.
- Gamma is the second derivative of the option price with respect to the stock: it can be stated as $\Gamma = d^2C/dS^2$ or $\Gamma = d^2P/dS^2$.

Gamma

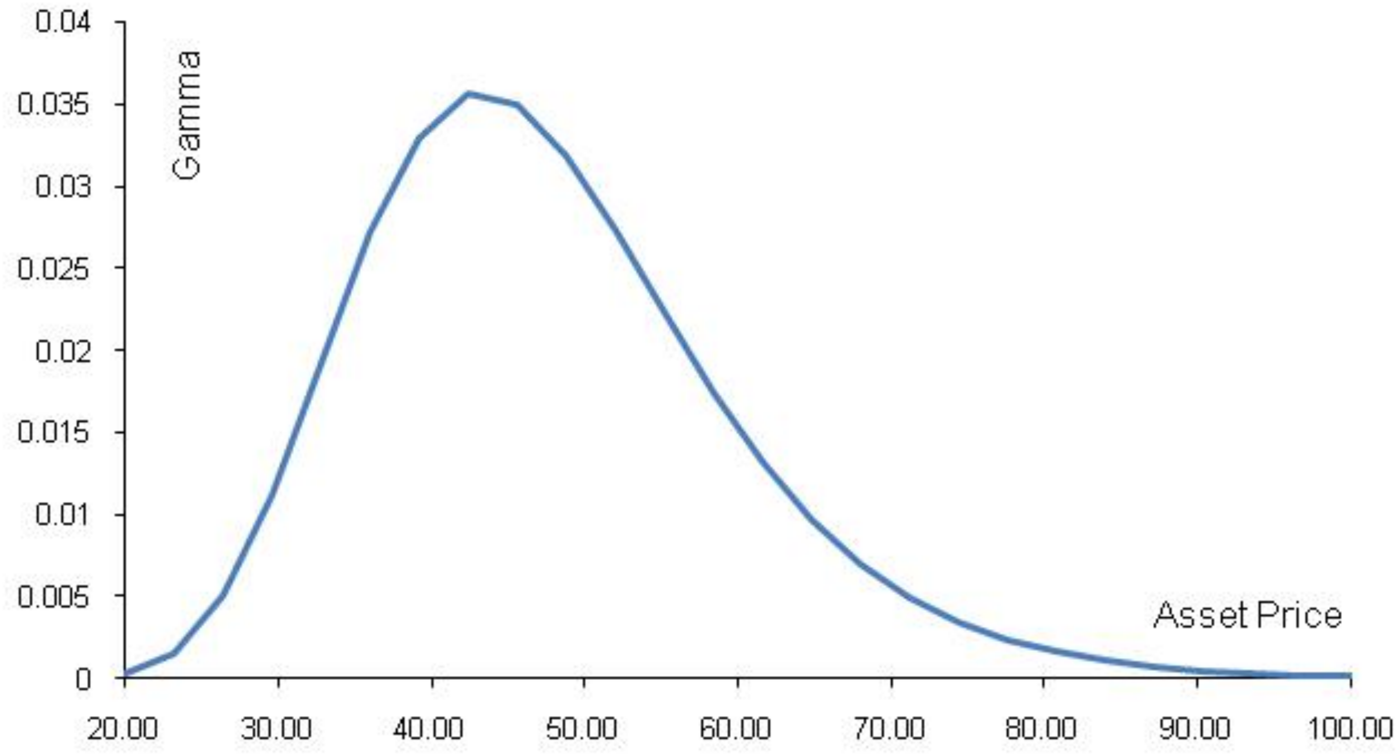
- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

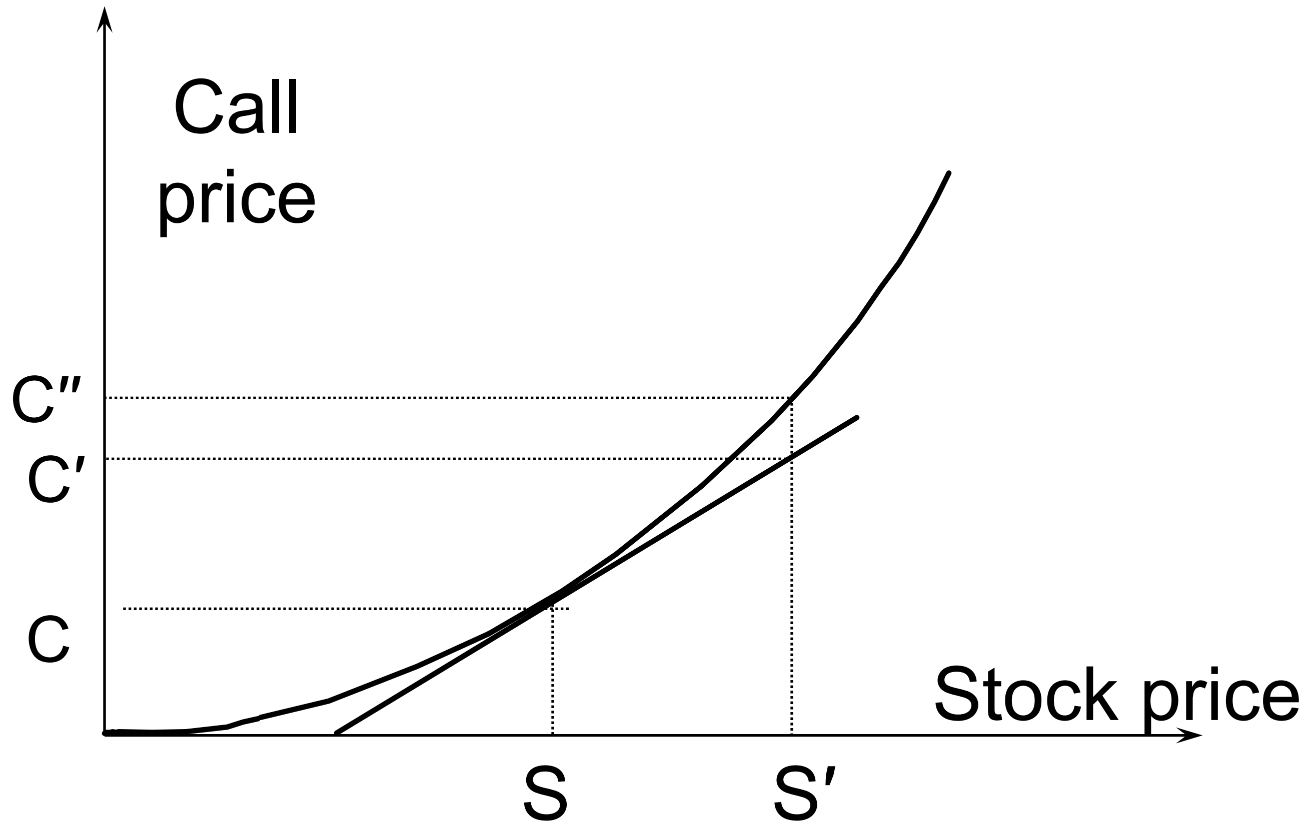
$$with\ N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- The formula is valid for both call and put options.

Gamma for Call or Put Option: K=50, σ = 25%, r = 5%, T = 1



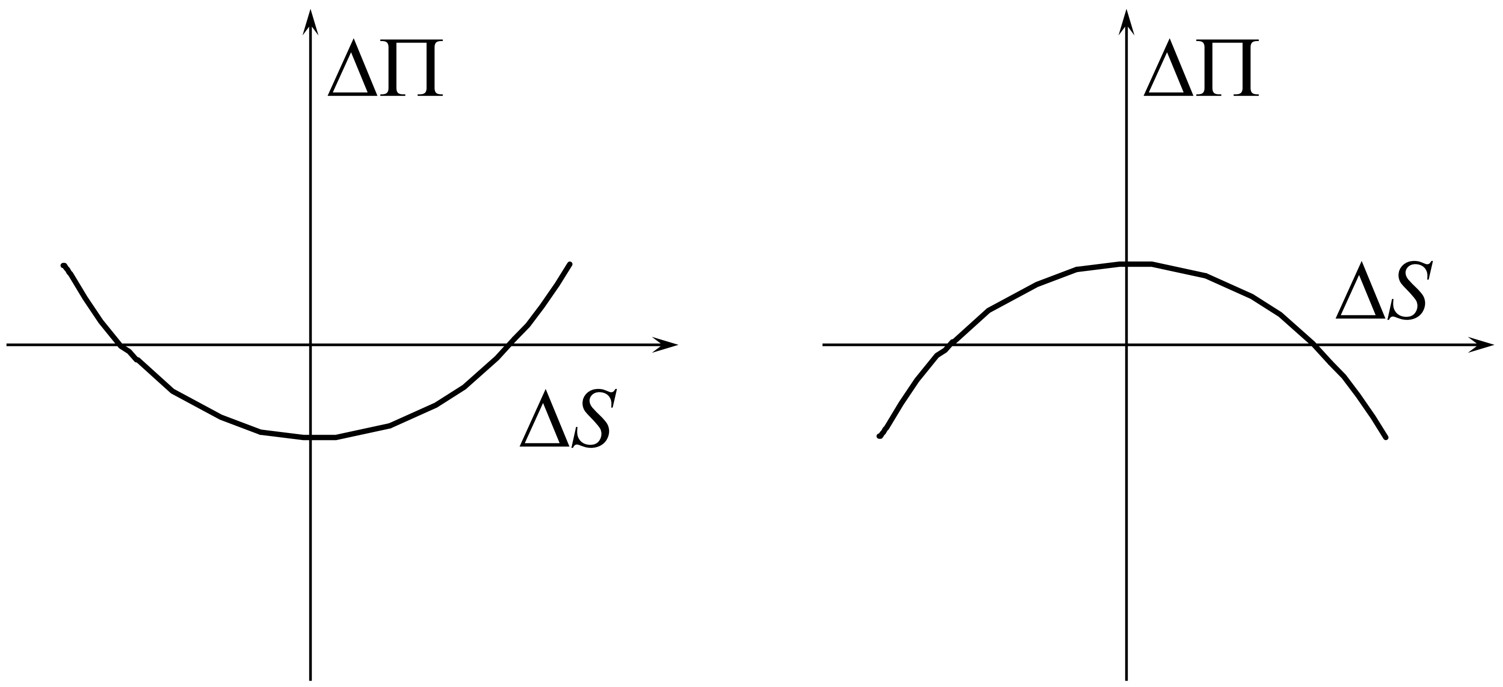
Gamma Addresses Delta Hedging Errors Caused By Curvature
(Figure 17.7, page 377)



Interpretation of Gamma

- For a delta neutral portfolio,

$$\Delta\Pi \approx \Theta\, \Delta t + ½\Gamma\Delta S^2$$



Positive Gamma

Negative Gamma

Relationship Between Delta, Gamma, and Theta

For a *portfolio* Π *of derivatives* on a non-dividend-paying stock, we have:

$$\Theta + rS_0\Delta + \frac{1}{2}\sigma^2 S_0^2\Gamma = r\Pi$$

$$or \quad \frac{\partial C}{\partial t} + rS_0\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 C}{\partial S^2} = rC \quad \textit{if the portfolio consists of one Call only}$$

Solving this partial differential equation for *C* (or *P* if a Put) yields the Black-Scholes pricing equation, by the way.

How to make a portfolio “Gamma Neutral”

- Recall that, just like for any other Greek (Delta,…), the Gamma of a portfolio Π is given by: $\Gamma_{\Pi} = n_1\Gamma_1 + n_2\Gamma_2 + n_3\Gamma_3 \ldots$
- One can thus make the total Gamma equal to zero with the appropriate addition of a certain number of options with a certain Gamma.

- As an example, assume your position is currently Delta neutral (Δ=0) but that Γ=-3,000. You however find a call option with Δ_C=0.62 and Γ_C=1.50
- You need $\Gamma_{\Pi} = (1)\Gamma_{\text{exisiting position}} + n_C\Gamma_C = 0$ i.e. need $-3{,}000 + n_C\,(1.50) = 0$.
- Buy n_C = 3,000/1.5 = **2,000 Call options** and now have a Gamma of zero.

- The only issue is that the addition of the call options shifted your delta.
- Your new portfolio delta is $\Delta_{\Pi} = (1)0 + 2{,}000\Delta_C = 2{,}000(0.62) = 1{,}240$.
- Therefore 1,240 shares of the underlying asset must be sold from the portfolio in order to keep it delta neutral. It is now <u>Delta-Gamma neutral</u>.

Vega (not an actual Greek letter, but it sounds like one: good enough)

- Vega (ν) is the rate of change of the value of a derivative (or a *derivatives portfolio*) with respect to the *volatility* of the underlying asset.

- Vega is an important measure because in practice, the volatility σ is not constant and changes over time.

- If Vega is highly positive or negative, the portfolio's value is very sensitive to small changes in volatility.

- If Vega is close to zero, volatility changes have almost no impact on the value of the portfolio.

Vega

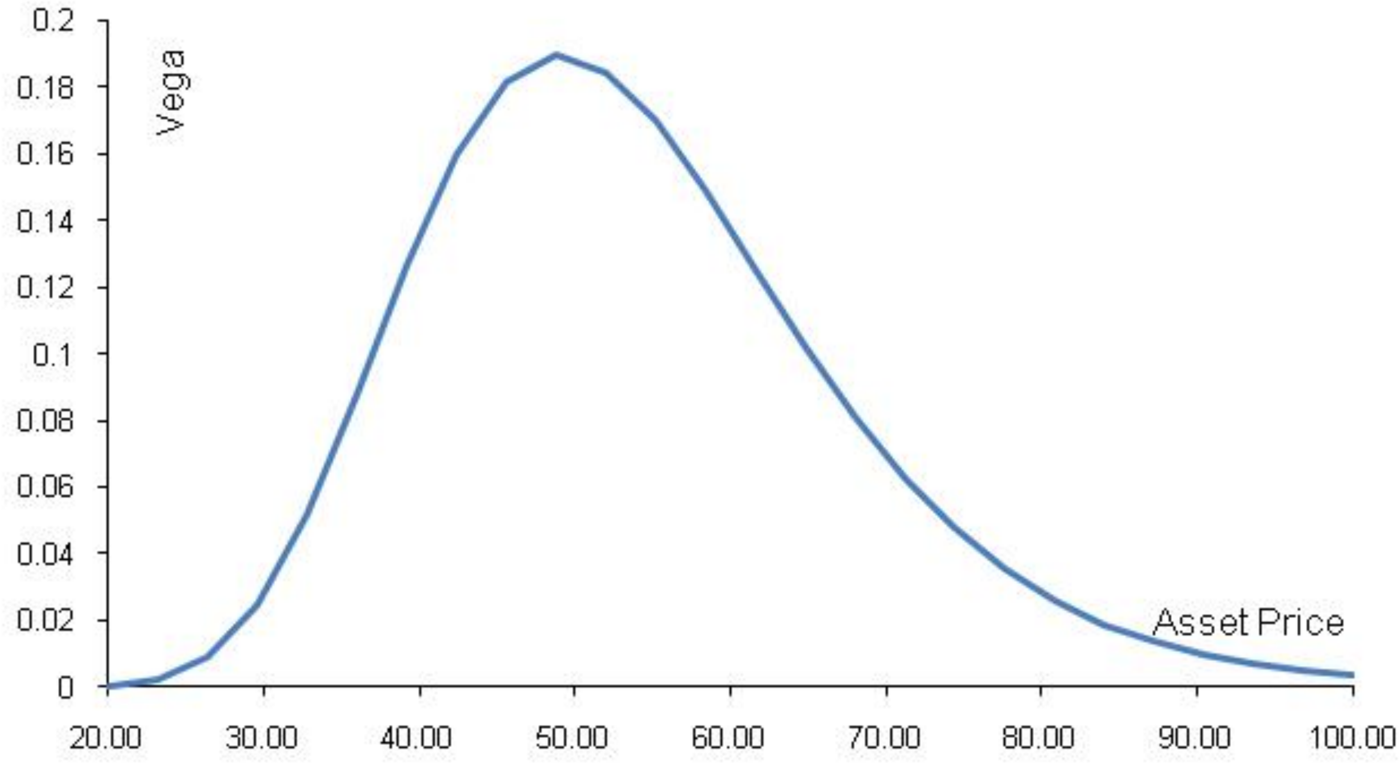
- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

$$\nu = S_0 \sqrt{T} N'(d_1)$$

$$\text{with } N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- The formula is valid for both call and put options.

Vega for a Call or Put Option: K=50, σ = 25%, r = 5%, T = 1



Managing Delta, Gamma, & Vega risk *all at once*

- We know that Delta can be changed by taking a position in the underlying asset.
- We also know that to adjust Gamma and Vega, it is necessary to take a position in an option or other derivative.
- However, if only one other derivative is added, either the Gamma risk or the Vega risk will be canceled, but not both at the same time (except by some coincidence).
- So we need to add ***two* new derivatives** to hedge all risks.

Example if we don't

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral?
Answer: Long 10,000 options, short 6,000 of the asset.

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral?
Answer: Long 4,000 options, short 2,400 of the asset.

Example if we do

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral *all at once*?

We solve

$$-5000 + 0.5w_1 + 0.8w_2 = 0$$
$$-8000 + 2.0w_1 + 1.2w_2 = 0$$

to get w_1 = 400 and w_2 = 6,000.
We therefore require long positions of 400 and 6,000 in option 1 and option 2.

However, because these additions result in an incremental positive delta of 400(0.6)+6,000(0.5)=3,240, we also need to take a *short position* of 3,240 in the asset in order to also make the portfolio delta neutral.

Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate.

- It is usually small and not a big issue in practice, unless the option is deep in-the-money and has a long horizon (discounting a larger cash flow over a longer horizon is more relevant then).

Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day.
- Whenever the opportunity arises, they improve gamma and vega.
- As the portfolio becomes larger, hedging becomes less expensive since the trading cost *per option* goes down.

Scenario Analysis

- In addition to monitoring risks such as delta, gamma, and vega, option traders often also conduct a scenario analysis.
- A scenario analysis involves computing the gains and losses on the portfolio over a specified period of time under a variety of different scenarios.
- Often the two main sources of risk looked as variables (the scenarios) are the underlying asset price and volatility.

Greek Letters for European Options on an Asset that Provides a (dividend) Yield at Rate *q* (Table 17.6, page 386)

Greek Letter	Call Option	Put Option
Delta	$e^{-qT}N(d_1)$	$e^{-qT}\left[N(d_1)-1\right]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $-qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Using Futures for Delta Hedging

- The delta of a futures contract on an asset paying a yield at rate q is $e^{(r-q)T}$, since we know that $F=Se^{(r-q)T}$.

- The position required in futures for delta hedging (instead of using the spot asset) is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot asset.

Example of Using Futures for Delta Hedging (instead of the spot)

- A portfolio of currency options held by a US bank can be made delta neutral with a short position of 458,000 pounds sterling (of the spot (currency) asset, if it were used).

- If the US riskless rate is 4% and the UK rate 7%, hedging for 9 months using a short position in futures contracts instead of shorting the spot currency would require shorting:

  $e^{-(0.04-0.07)x9/12}$ x 458,000 or £468,422 in futures contracts.

- Since each futures contract is for £62,500 the number of contracts to be shorted is 468,422/62,500 = 7.49 contracts therefore rounded to 7 contracts.

Hedging vs. Creation of an Option Synthetically

- When we are hedging we take positions that offset delta, gamma, vega, etc…
- When we create an option synthetically we take positions that match delta, gamma, vega, etc…

Portfolio Insurance

- In October of 1987, many portfolio managers attempted to create a put option on a portfolio synthetically.

- This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option.

Portfolio Insurance (continued)

- As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased.
- As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold.
- The strategy did not work well on October 19, 1987 because since everyone was doing the same thing, liquidity became an issue.
- Additionally, investors that anticipated the portfolio insurers reaction sold their positions as well, exacerbating the problem and precipitating the price decrease, making the crash worse.