

Multifactor Term Structure Models

A. Limitations of One-Factor Models

1. Returns on bonds of all maturities are perfectly correlated.
2. Term structure (and prices of every other derivatives) are uniquely determined by the level of the spot rate.
3. Most Common Term Structure Movements - PCA (Principal Components Analysis)

One convenient way to analyze the term structure is to use principal components analysis or factor analysis.

Let $y(t, \tau)$ represent a zero coupon bond with time to maturity τ .

Assume that we are given the yields of Treasuries zero coupon bonds with maturities 3 and 6 months and 1,2,3,4,5,7,10 and 30 years. That is, we have 10 different maturities.

$y(t, \tau_1)$ = yield of the 3-month zero coupon bond at time t

$y(t, \tau_2)$ = yield of the 6-month zero coupon bond at time t

⋮

$y(t, \tau_{10})$ = yield of the 10-year zero coupon bond at time t

Assume that we describe the changes on these yields on the following way:

$$\Delta y(t, \tau_i) = v_1(\tau_i) \Delta a_1(t) + v_2(\tau_i) \Delta a_2(t) + \dots + v_{10}(\tau_i) \Delta a_{10}(t)$$

where

$a_j(t)$ are 10 **independent** random variables and $\Delta a_j(t)$ is the change on a_j at time t . $a_j(t)$ is called a principal component or factor.

$v_j(\tau_i)$ is the loading of the yield of the bond with time to maturity τ_i on the j^{th} factor.

The equation above implies that if the principal component $a_j(t)$ increases by 1bp's then $\Delta a_j(t) = 1/10000$ then the yield $y(t, \tau_i)$ changes by $v_1(\tau_i)$ basis points.

The equation above also implies that the variance covariance matrix of the changes in yields is given by:

$$\text{cov}(\Delta y(t, \tau)) = v \times \Lambda \times v'$$

where v is a 10×10 matrix with the i -th row given by $v_1(\tau_i), v_2(\tau_i), \dots, v_{10}(\tau_i)$ and Λ (the matrix of eigenvalues of $\text{cov}(\Delta y(t, \tau))$) is a 10×10 matrix with the diagonal elements equal to the variances of the factors.

The importance of the factor is measured by the relative variance of the factor. The variances of the factors have the property that they add up to the total variance of the data (that is, the sum of the variance of the observations on the three-month rate, six-month rate and so on...). Therefore the importance of the factor is measured by:

$$\frac{\text{Variance of the factor}}{\text{Total Variance of the Data}}$$

Litterman and Scheinkman, Journal of Fixed Income, 1991, did the exercise above. They found the matrix v (factor loadings) and the matrix Λ for the US rates.

- The first factor is a roughly parallel shift (83.1% of variation explained). This motivates the name of the first factor as the level factor.
- The second factor is a twist (10% of variation explained). The second factor is called the “slope” of the curve.
- The third factor is a bowing (2.8% of variation explained). The third factor is usually called the “curvature” factor.

B. Multifactor CIR Model

1. Each factor has the CIR dynamics

a. $dF_i = \beta_i (\alpha_i - F_i) dt + \sigma_i \sqrt{F_i} dZ_i$, $i=1, \dots, K$ and the spot rate is $r = \sum_{i=1}^K F_i$.

b. Then zero-coupon bond prices are given by:

$$B(F_1, \dots, F_K, t, T) = a_1(t, T) \dots a_K(t, T) e^{-F_1 b_1(t, T) - \dots - F_K b_K(t, T)}$$

c. The yields are still affine functions of the factors:

$$\begin{aligned} y(F_1, \dots, F_K, t, T) &= -\frac{1}{T-t} \text{Ln}(B(F_1, \dots, F_K, t, T)) \\ &= -\sum_{i=1}^K \frac{\text{Ln}(a_i(t, T))}{T-t} + \sum_{i=1}^K \frac{b_i(t, T)}{T-t} F_i \\ &\equiv A + BF \end{aligned}$$

d. The estimation of the model is usually made as in Chen and Scott, "Multifactor CIR Models of the Term Structure", Journal of Fixed Income:

Assume that the prices of K zero-coupon bonds are observed. We can then perform an inversion of the pricing equations to obtain a time series of the latent state variables, which is used to estimate the model parameters by maximum likelihood.

$$y(F_1, \dots, F_K, t, T) = A + B F_t + \varepsilon_t$$

The likelihood of the observed rates can be computed from the likelihood of the factors by change in variables.

e. Duffie and Singleton (1997) estimate a 2-factor CIR model using Libor and Swap rates data in a way similar to the one explained above.

C. General Affine Models

a. Duffie and Kan, “Mathematical Finance” 1996, generalize the CIR and Vasicek models to a class of models called affine.

b. The zero-coupon bond prices in affine models are given by:

$$B(F_1, \dots, F_K, t, T) = a_1(t, T) \dots a_K(t, T) e^{-F_1 b_1(t, T) - \dots - F_K b_K(t, T)}$$

that is, the yields of zero-coupon bonds is given by affine functions of the factors F .

c. To define an arbitrage-free affine term-structure model, they prove that it is necessary to:

1. $r = \delta_0 + \sum_{i=1}^K \delta_i F_i$
2. The drift of the state variables under the risk neutral measure Q must be an affine function of the state variables.
3. The diffusion of the state variables must be given by $\Sigma \times \sqrt{S_t}$ where Σ is a matrix of constants and S_t is a diagonal matrix of affine functions of the state variables F .

d. Dai and Singleton, JF 2002 empirically analyze affine models. The analysis of affine models is complicated by the fact that it is not possible to build a K factor affine model that nests all the other K factor models.

So they classify affine models in terms of the number of factors that affect the volatility of yields. And they define the notation $A_m(n)$ where n is the number of factors in the model and m is the number of factors that affect the volatility of yields. For instance, a 3-factor CIR model is within the family of $A_3(3)$ models and a 3-factor Vasicek model is within the family of $A_0(3)$ models.

Within each sub-family of affine models, they give conditions to find the maximal model, that is the model in which all the models of the $A_m(n)$ class are nested.

D. Quadratic Term Structure Models

1. Model the pricing kernel directly

a. Recall that for a zero-coupon bond:

$$B(t, T) = E^* \left[\exp \left\{ - \int_t^T r_u du \right\} \middle| r_t = r \right]$$

b. Define the *pricing kernel* to be $M(\cdot)$ such that:

$$\frac{M(T)}{M(t)} = \exp \left\{ - \int_t^T r_u du \right\}$$

c. Model the dynamics of $M(\cdot)$ directly:

- Assume that there K factors driving $M(\cdot)$
- And that $M(\cdot)$ has the form:

$$M(t) = \exp \{ \text{quadratic}(F_i) \}$$

- Where F_i 's follow Ornstein-Uhlenbeck processes:

$$dF_i = -\gamma_i F_i dt + \sigma_i dZ_i$$

2. This model can be solved explicitly for bond and other derivative prices

a. All we need is the expected value of $M(T)$ given the values of the factors at t . This can be done because:

- Ornstein-Uhlenbeck implies Gaussian distributions
- There are formulas for the expected value of the exponential of a Gaussian random variable, or its square.

b. Reference: Constantinides, G.M., "A Theory of the Nominal Term Structure of Interest Rates," *Review of Financial Studies*, 1992, 531-552.

3. This model can also be put in state-space form to infer the factors from the observed yields. Reference: Zheng, C., University of Chicago PhD Dissertation, 1994 and Ahn et. Al., RFS 2002.

E. Unobservable Factors: Use Kalman Filter

1. The yields on bonds with a variety of maturities are observable, but we may want to avoid pre-specifying the factors.

2. Put the model in *state-space form*:

a. We can filter the values of the factors at each point in time based on the observed yields:

$$\begin{cases} Y_t = A + BF_t + \eta_t \\ F_t = \Gamma + PF_{t-1} + \varepsilon_t \end{cases}$$

- Y_t is a vector of yields at t of bonds of different maturities
- F_t is the vector containing the values of the K factors at time t
- A , B , Γ and P are matrices of constant parameters. The parameters in B are called the factor loadings.
- η_t and ε_t are random disturbances

b. The Kalman filter produces estimates of the parameters as well as estimates of F_t at every date.

c. Reference: For the general state-space approach: Harvey, A.C. , "Time-Series Models," MIT Press, 1993, Chapter 4.