Numerical Methods in Option Pricing (Part II)

II. Binomial Trees

A. Cox-Ross-Rubinstein Tree (Black-Scholes setting)

a. We construct the tree to replicate the risk-neutral process of the underlying asset:

\[
\begin{align*}
S_{n+1}^{m-n-1}f_{n+1}^m & \quad q \quad S_{n+1}^{m-n}f_{n+1}^m & \quad 1-q \\
S_{n}^{m-n}f_{n}^m & \quad 1-q \quad S_{n}^{m-n+1}f_{n-1}^m & \quad 1-q
\end{align*}
\]

Where \( f_n^m \) is the value of the derivative at node \((n,m)\) with \(n\) the number of times the asset went up and \(m\) representing the time period.
b. We can approximate the diffusion
dS = rSdt + \sigma SdZ_t^Q
by:
\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} \\
    d &= e^{-\sigma \sqrt{\Delta t}} \\
    q &= (a - d) / (u - d) \quad \text{where } a = e^{r \Delta t}
\end{align*}
\]

c. Starting from the final period, now use risk-neutral valuation backward, period by period:
\[
f_n^m = \frac{1}{1 + r \Delta t} \left[ q f_{n+1}^{m+1} + (1 - q) f_n^{m+1} \right]
\]
d. The value of the derivative in the final period is given by the payoff function g(.). In the case of the European call option, for instance, the payoff at time $M \Delta t$ is:
\[
f_n^M = \max(0, S_u^n d^{M-n} - K), \ n=0,1,...,M.
\]
e. To realistically approximate the derivative price, let $\Delta t$ go to 0.

f. The advantage of such a tree is to have constant values for u, d, and q. They do not depend on the time period (m) nor do they depend on the stock level (n).
B. How to Estimate the “Greeks”

1. Delta (hedge ratio)
   a. We know that \( \Delta = \frac{\partial f}{\partial S} \) (change in option value for a change in S)
   b. Delta can thus be estimated at node \((n,m)\) by:

   \[
   \Delta_n^m = \frac{f_n^{m+1} - f_n^{m-1}}{S_u^{n+1}d^{m-n-1} - S_u^{n-1}d^{m-n+1}} \quad \text{(we want it centered around } f_n^m) \]

2. Other Hedge Ratios
   a. The second derivative or curvature measure is \( \Gamma = \frac{\partial^2 f}{\partial S^2} \)

   In other words, \( \Gamma \) is the change in the \( \Delta \), or the change in the slope. Graphically, we have:
The first slope or derivative is \( (f_2 - f_1)/\Delta x \) and the second slope or derivative is \( (f_3 - f_2)/\Delta x \). Notice that the centers/midpoints of the two slopes/lines are \( \Delta x \) apart. Therefore, the change in the slope over a \( \Delta x \) interval can be expressed as:

\[
\Gamma = \frac{\text{second slope} - \text{first slope}}{\Delta x} = \frac{f_3 - f_2}{\Delta x} - \frac{f_2 - f_1}{\Delta x} = f_3 - 2f_2 + f_1
\]

Therefore, we have:

\[
\Gamma_n^m = \frac{f_n^{m+1} - 2f_n^m + f_n^{m-1}}{(\Delta x)^2}
\]

b. The change in the option value over time is \( \Theta = \frac{\partial \Theta}{\partial t} \), and so

\[
\Theta_n^m = \frac{f_n^{m+1} - f_n^{m-1}}{2\Delta t}
\]

(so that we are comparing two points where the stock level is the same and the only thing that differs is time)

c. The change in the option value with respect to the risk-free rate is \( \rho = \frac{\partial \rho}{\partial r} \). Here we need to construct two trees: one tree for \( r \), and another for \( r + \Delta r \). Estimate \( \rho_n^m = \frac{f_n^m(r + \Delta r) - f_n^m(r)}{\Delta r} \).

d. The change in the option value with respect to the volatility is \( \text{Vega} = \frac{\partial \text{Vega}}{\partial \sigma} \). We also need to construct two trees here: one tree for \( \sigma \), and another for \( \sigma + \Delta \sigma \). Then estimate

\[
\text{Vega}_n^m = \frac{f_n^m(\sigma + \Delta \sigma) - f_n^m(\sigma)}{\Delta \sigma}
\]
C. How to Adjust for Payments by the Underlying Asset (Options on: Index, Futures, and Currencies)

a. Case of an Index Option (ex: S&P 500 options):
   Replace $r$ in $a=e^{r\Delta t}$ by $r-\delta$ ($\delta=$dividend yield) in the formula for the risk-neutral probability $q$.
   So $a=e^{(r-\delta)\Delta t}$.

b. Case of an Option on Futures:
   Unlike a stock, an index, or a currency, futures contracts are simply agreements between two parties and thus do not require any payment upfront (ignoring margin requirements and such). Since it costs nothing to enter such a contract, the risk-neutral dynamics of the futures $F$ must be: $dF^*=\sigma F^*dZ$ (no drift, or $\delta=r$ in the expressions above). So $a=1$.

c. Case of a Currency Option:
   Example: 1-year foreign currency European call option on the British Pound. The US risk-free rate is denoted by $r_{US}$ and the UK risk-free rate is denoted by $r_{UK}$.

   • The underlying factor is the exchange rate and its dynamics are given by: $de^*=(r_{US} - r_{UK})e^*dt + se^*dW$.

   • It is mathematically the same thing as having a dividend payout, with $r=r_{US}$ and $\delta=r_{UK}$. Thus we have $a=e^{(r_{US}-r_{UK})\Delta t}$ here.

d. Finally notice that the discounting still takes place at the same rate $r$, so $C=[qC_u+(1-q)C_d]/[1+r\Delta t]$. The only difference is that we have adjusted $q$ to reflect $r-\delta$ in “a”.
III. Numerical Methods for PDEs

A. Types of PDEs and Terminology

a. Example of a simple Pricing PDE we have seen for a traded asset:

\[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} (r - \delta)S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 - rf = 0 \]

b. Example of a simple Pricing PDE we have seen in the case of a non-traded factor:

\[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} (m - \lambda s)F + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} s^2 F^2 + h - rf = 0 \]

c. General form for Second Order Linear PDEs:

\[ a(S,t) \frac{\partial^2 f}{\partial S^2} + b(S,t) \frac{\partial^2 f}{\partial S \partial t} + c(S,t) \frac{\partial^2 f}{\partial t^2} + d(S,t) \frac{\partial f}{\partial S} + e(S,t) \frac{\partial f}{\partial t} + g(S,t) f + h(S,t) = 0 \]

The discriminant \( \Delta \) is \( b^2 - 4ac \)

- If \( \Delta \) is positive, the equation is said to be hyperbolic.
- If \( \Delta \) is negative, the equation is said to be elliptic (never happens in finance).
- If \( \Delta = 0 \), the equation is said to be parabolic (the type we encounter in finance where \( b = c = 0 \)).
- Notice that this way of classifying is local: an equation may be elliptic at some \((S,t)\) and parabolic at others.
d. **Backward vs. Forward PDEs:**

\[
\frac{\partial f}{\partial t} + a(S,t) \frac{\partial^2 f}{\partial S^2} + d(S,t) \frac{\partial f}{\partial S} + g(S,t)f + h(S,t) = 0
\]

- If \(a > 0\), the equation is backward.
- If \(a < 0\), the equation is forward.

**B. General Tricks to Find a Closed-Form Solution**

Before resorting to numerical methods, it is worth first checking whether a closed-form solution exists. Looking for a PDE of the same form in a book referencing PDEs with known solutions at the Math library is generally a good way to go, but here are a few general tips that may help you solve the equation on your own.

a. Apply successive changes of variables to attempt to transform the original equation into the heat equation that has a known solution. Example: \(x = \log(S)\) or \(x = \log(S/K)\), \(\tau = (T-t)/(\sigma^2/2)\).

b. Decrease the number of parameters to get the equation to a simpler form, for example: \(\kappa_1 = r/(\sigma^2/2)\) and/or \(\kappa_1 = (r-\delta)/(\sigma^2/2)\).

c. Try solutions in the form \(V(x,\tau) \equiv e^{\alpha x + \beta \tau} U(x,\tau)\), or \(V(x,\tau) \equiv \tau^\gamma W(x/\tau^\delta)\) and choose the parameters \(\alpha\) and \(\beta\) or \(\gamma\) and \(\delta\) to simplify the equation.
C. PDEs Numerical Approximation

1. First Derivatives

   a. These limits are equivalent (but centered differently):

   \[
   \frac{\partial f(S, t)}{\partial t} = \lim_{\Delta t \to 0} \frac{f(S, t + \Delta t) - f(S, t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(S, t) - f(S, t - \Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(S, t + \Delta t) - f(S, t - \Delta t)}{2\Delta t}
   \]

   The equivalent for S would be:

   \[
   \frac{\partial f(S, t)}{\partial S} = \lim_{\Delta S \to 0} \frac{f(S + \Delta S, t) - f(S, t)}{\Delta S} = \lim_{\Delta S \to 0} \frac{f(S, t) - f(S - \Delta S, t)}{\Delta S} = \lim_{\Delta S \to 0} \frac{f(S + \Delta S, t) - f(S - \Delta S, t)}{2\Delta S}
   \]

   b. Thus we can approximate the derivative by calculating the right-hand side of the equations above for a small \( \Delta t \) or a small \( \Delta S \).

   • Forward Difference for time:

   \[
   \frac{\partial f(S, t)}{\partial t} \approx \frac{f(S, t + \Delta t) - f(S, t)}{\Delta t}
   \]
• Backward Difference for **time**:

\[
\frac{\partial f(S,t)}{\partial t} \approx \frac{f(S,t) - f(S, t - \Delta t)}{\Delta t}
\]

• Central Difference for **time**:

\[
\frac{\partial f(S,t)}{\partial t} \approx \frac{f(S, t + \Delta t) - f(S, t - \Delta t)}{2\Delta t}
\]

• Forward Difference for the **stock price**:

\[
\frac{\partial f(S,t)}{\partial S} \approx \frac{f(S + \Delta S, t) - f(S, t)}{\Delta S}
\]

• Backward Difference for the **stock price**:

\[
\frac{\partial f(S,t)}{\partial t} \approx \frac{f(S, t) - f(S - \Delta S, t)}{\Delta S}
\]

• Central Difference for the **stock price**:

\[
\frac{\partial f(S,t)}{\partial t} \approx \frac{f(S + \Delta S, t) - f(S - \Delta S, t)}{2\Delta S}
\]
2. Second Derivatives

a. These limits are equivalent (but centered differently):

$$\frac{\partial^2 f(S,t)}{\partial S^2} = \lim_{\Delta S \to 0} \frac{\partial f(S + \Delta S, t)}{\partial S} \frac{\partial^2 f(S, t)}{\partial S^2} \Delta S$$

$$= \lim_{\Delta S \to 0} \frac{\partial f(S, t)}{\partial S} - \frac{\partial f(S - \Delta S, t)}{\partial S} \Delta S$$

$$= \lim_{\Delta S \to 0} \frac{\partial f(S + \Delta S, t)}{\partial S} - \frac{\partial f(S - \Delta S, t)}{\partial S} 2 \Delta S$$

This is because:
b. To obtain the symmetric central difference for the second derivative, we use the following:

\[
\frac{\partial^2 f(S,t)}{\partial S^2} \approx \frac{\partial f(S + \Delta S,t)}{\partial S} - \frac{\partial f(S,t)}{\partial S} \approx \frac{\left\{ f(S + \Delta S,t) - f(S,t) \right\} - \left\{ f(S,t) - f(S - \Delta S,t) \right\}}{\Delta S^2}
\]

(Notice that for the discretization of the first derivatives \( \frac{\partial f(S + \Delta S,t)}{\partial S} \) and \( \frac{\partial f(S,t)}{\partial S} \) we went \textit{backward} for both in order to be consistent, i.e. when the starting point is \( S + \Delta S \) we go to \( S \), and when the starting point is \( S \) we go to \( S - \Delta S \).)

Also note that if we had used \( \frac{\partial f(S,t)}{\partial S} \) and \( \frac{\partial f(S - \Delta S,t)}{\partial S} \) instead and discretized the first derivatives by going \textit{forward}, we would have obtained the exact same result.

Finally, the last expression yields:

\[
\frac{\partial^2 f(S,t)}{\partial S^2} \approx \frac{f(S + \Delta S,t) - 2f(S,t) + f(S - \Delta S,t)}{(\Delta S)^2}
\]

This result is consistent with what we had obtained before on page 4 of the notes. It can also be shown that the central approximation is an order of magnitude more precise than either the forward or backward difference.
D. Finite Difference Grid

1. Construct an Equally-Spaced Grid

a. The PDE that needs to be solved is:

\[
\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial S} + b \frac{\partial^2 f}{\partial S^2} + h - rf = 0
\]

for \( \underline{S} < S < \overline{S} \) and \( t > 0 \)

Subject to:

- \( f(S,0) = f_0(S) \) [initial/boundary condition]

(Note that \( a, b, \) and \( h \) can be functions of \( S \) and \( t \)).

In option pricing, we usually have a terminal condition instead of an initial one. We can switch between the two with a change of variable from \( t \) to \( \tau = T - t \). This change of variable therefore changes \( \frac{\partial f}{\partial t} \) to \( -\frac{\partial f}{\partial \tau} \).

The PDE therefore becomes:

\[
-\frac{\partial f}{\partial \tau} + a \frac{\partial f}{\partial S} + b \frac{\partial^2 f}{\partial S^2} + h - rf = 0
\]
b. **Construct a grid in time and state (state = stock price)**

The goal is to solve for the various values of the derivative price \( f \) on the various points of the grid, where we define 
\[
f_n^m = f(n\Delta S, m\tau).
\]
Note that we will interpolate between grid points, and note that the time on the grid is defined as time-to-expiration/maturity.
2. Implementation

There are different types of finite-difference approximations. The specific type we chose to implement for the first and second-order derivatives that appear in the Partial Differential Equation will determine the nature of the algorithm we use.

a. Explicit

b. Implicit
   - Fully Implicit
   - Crank-Nicolson