American Options

A. Optimality of Early Exercise

1. American Call Options without Dividends

a. C designs a European Call, and c designs an American call. P designs a European Put, and p designs an American put.

b. Let's first show that for a stock paying no dividends between now and expiration, we have:

$$c(S,t) \geq C(S,t) \geq \max[0, S-KB(t,T)]$$

Proof:

• Let B(t,T) be the price at time t of a zero-coupon bond paying \$1 at time T. This means that the value KB(t,T) is the present value at time t of the exercise price K.

• Consider the following portfolio 1: purchase one call option C(S,t) and KB(t,T) bonds. Now also consider the following portfolio 2: Buy the stock for the price S_t.

The values today (t) and at maturity (T) are given in the following table:

	Portfolio 1: buy one call and KB bonds	Portfolio 2: buy one share of stock
Current Value	$C(S_t,t) + KB(t,T)$	S_t
Value at T if $S_T>K$	$[S_T-K] + K = S_T$	S_T
Value at T if $S_T<K$	K	S_T

• At maturity (T), we can see that portfolio 1 dominates portfolio 2 for all possible terminal values of S_T. Thus the value of portfolio 1 <u>today</u> must be higher than the value of portfolio 2. In other words, $C(S,t) + KB(t,T) \geq S$. Therefore we have $C(S,t) \geq S - KB(t,T)$. And since an option must always have a value above zero (since one can always walk away), we have:
$C(S,t) \geq \max[0, S\text{-}KB(t,T)]$.
The same reasoning applies to the American call $c(S,t)$ which must also be at least as valuable as the European call, therefore:
$c(S,t) \geq C(S,t) \geq \max[0, S\text{-}KB(t,T)]$

c. Define the insurance value of the call to be: $IC(S,t)$ so that we have:

$$c(S,t) = S - KB(t,T) + IC(S,t)$$

• Since we know that $C(S,t) + KB(t,T) \geq S$, we thus know that $IC\geq 0$.

• Rearrange the expression and get:

$$\underbrace{c(S,t)-\{S-K\}}_{\substack{\text{net loss if}\\ \text{exercise at t}}} = \underbrace{K-KB(t,T)}_{\substack{\text{time value of money}\\ \text{on K}}} + \underbrace{IC(S,t)}_{\substack{\text{insurance value}\\ \text{of the call}}}$$

• The left-hand side is what you loose if you exercise at t: the value of the call is lost (you didn't sell the option) but you get S-K (S is gained and K is paid).

• If it is not optimal to exercise early, we have: c(S,t) > S - K

• The equation shows that two elements impact the decision of exercising early: the loss associated with paying K at time t instead of T (hence you lose the time value of money on K), and the loss of insurance value that the call provides you.

• Since IC≥0 and K>KB, exercising an American call option early implies a loss on both these elements, and thus this shows that it is never optimal to exercise a non-dividend paying American call option early. Finally, since we would never use the option to exercise when there are no dividends, the price of an American call option is the same as the price of a European Call option.

2. American Put Option without Dividends

a. Using the same method as in the last section, we can show that:

$$p(S,t) \geq P(S,t) \geq \max\ [KB(t,T)-S,0]$$

Proof:

• Let B(t,T) be the price at time t of a zero-coupon bond paying \$1 at time T. This means that the value KB(t,T) is the present value at time t of the exercise price K.

• Consider the following portfolio 1: purchase one put option P(S,t) and sell KB(t,T) bonds. Now also consider the following portfolio 2: Short-sell the stock for the price S_t.

The values today (t) and at maturity (T) are given in the following table:

	Portfolio 1: buy one put and sell KB bonds	Portfolio 2: short-sell one share of the stock
Current Value	$P(S_t,t) - KB(t,T)$	$-S_t$
Value at T if $S_T>K$	$0 - K = -K$	$-S_T$
Value at T if $S_T<K$	$K - S_T - K = - S_T$	$-S_T$

• At maturity (T), we can see that portfolio 1 is either better than portfolio 2 (when $S_T>K$) or the same (when $S_T<K$). Thus the value of portfolio 1 today must be higher than the value of portfolio 2.
In other words, $P(S,t) - KB(t,T) \geq - S$.
Therefore we have $P(S,t) \geq KB(t,T) - S$
And since an option must always have a value above zero (since one can always walk away), we have: $P(S,t) \geq \max(KB(t,T)-S,0)$

The same reasoning applies to the American put p(S,t) which must also be at least as valuable as the European put, therefore:
$p(S,t) \geq P(S,t) \geq \max\,[KB(t,T)-S,0]$

b. If we define the insurance value of an American put option as $IP(S,t) \geq 0$ such that:

$$p(S,t) = KB(t,T) - S + IP(S,t)$$

• The expression can be re-written as:

$$\underbrace{p(S,t) - \{K - S\}}_{\substack{\text{net loss if}\\ \text{exercise at t}}} = -\underbrace{\{K - KB(t,T)\}}_{\substack{\text{time value of money}\\ \text{on K}}} + \underbrace{IP(S,t)}_{\substack{\text{insurance value}\\ \text{of the put}}}$$

• As in the case of the call option, we have two factors determining the possible early decision: exercising implies receiving K at t instead of T (so you gain the time value on it) but as with the call, exercising also means that you are losing the insurance value that the option provides. Now, unlike the case of the call, here these two factors are going in two opposite directions, and so the sign of the expression is not known for sure.

• If the right-hand side turns out to be negative, we can then see that "the loss from exercising is negative", hence we should exercise.

3. American Call Option with Dividends

a. Assume a dividend D is paid at time τ, with $t<\tau\leq T$.

	Portfolio 1: buy one call and $KB(t,T)$ of bonds + $DB(t,\tau)$ of bonds	Portfolio 2: buy one share of stock
Current Value	$C(S_t,t) + KB(t,T) + DB(t,\tau)$	S_t
Value at T if $S_T>K$	$[S_T-K] + [K+D/B(\tau,T)]$ $= S_T + D/B(\tau,T)$	$S_T + D/B(\tau,T)$
Value at T if $S_T<K$	$K + D/B(\tau,T)$	$S_T + D/B(\tau,T)$

• Following the same reasoning as earlier, we can see that $C(S_t,t) + KB(t,T) + DB(t,\tau) \geq S$
and: $c(S,t) \geq C(S,t) \geq S - [KB(t,T) + DB(t,\tau)]$

• Note that $DB(t,\tau)$ is the present value at t of the dividend D paid at τ.

• For r=constant, we have $B(t,\tau)=e^{-r(\tau-t)}$ and we have $DB(t,\tau) = D\, e^{-r(\tau-t)}$.

• In the case of multiple dividends or continuous payments, replace $D/B(\tau,T)$ by the discounted cumulative value of the payments.

b. Let the insurance value of the call be $IC(S,t) \geq 0$ such that:

$$c(S,t) = S - [KB(t,T) + DB(t,\tau)] + IC(S,t)$$

• Rearrange the expression and get:

$$\underbrace{c(S,t) - \{S - K\}}_{\substack{\text{net loss if} \\ \text{exercise at t}}} = \underbrace{K - KB(t,T)}_{\substack{\text{time value of money} \\ \text{on K}}} + \underbrace{IC(S,t)}_{\substack{\text{insurance value} \\ \text{of the call}}} - \underbrace{DB(t,\tau)}_{\substack{\text{present value at t} \\ \text{of dividend paid at } \tau}}$$

• Note that instead of having two factors influencing the exercise decision, we now have three: exercising early implies the loss of time value of K, the loss of insurance value provided by the call, but the gain from receiving the dividends at τ.

• Therefore the early exercise of a call is optimal only if the present value of dividends is large enough to offset the loss of the insurance value of the call and the loss of the time value of K.

• Recall that we have $t<\tau<T$ and so the present value of the dividends is the larger the closer we are to τ. Thus the optimal exercise time, if it is optimal to early exercise, will be right before the dividend date.

• Note that if you were to early exercise any time prior to that date, you would be losing more time value on K and losing the insurance value of the option for a longer period of time. This thus would not be optimal.

4. American Put Options with Dividends

a. Using the same methodology as for the call option with dividends, we can derive:

$$\underbrace{p(S,t)-\{K-S\}}_{\substack{\text{net loss if}\\ \text{exercise at t}}}=-\underbrace{\{K-KB(t,T)\}}_{\substack{\text{time value of money}\\ \text{on K}}}+\underbrace{IP(S,t)}_{\substack{\text{insurance value}\\ \text{of the call}}}+\underbrace{DB(t,\tau)}_{\substack{\text{present value at t}\\ \text{of dividends paid at }\tau}}$$

b. In this case, exercising early implies gaining the time value of K, losing the insurance feature provided by the option, and finally losing the present value of the dividends.

- Notice that one aspect of minimizing some of the loss from exercising comes from having a small present value of dividends [DB(t,τ)]. This means we need either a long period of time between now and the payment of D, and/or small dividends D.
- Contrary to call options where early exercise can only happen right before the payment of dividends, early exercise in the case of put options can happen anytime.

5. Some Exercise Rules

a. If it is optimal to exercise a call option with a given strike price and time-to-expiration, then it is also optimal to exercise call options with lower strike prices and a shorter time to expiration. The reasons are:

• The time value of money on K (which you would be losing by exercising and thus hope to minimize) is smaller.

• The insurance value of the option (which you would also be losing by exercising, hence which you would also be trying to minimize) is also smaller because if the strike price is lower, this option is “more in-the-money” and therefore less likely to end up out-of-the-money (where the insurance feature of the option is useful).

b. Similarly, put options with larger strike prices and a longer time to expiration should be exercised.

B. American Options: Free Boundary Problem

1. European Put Option PDE:

a. $\frac{\partial P}{\partial t}+\frac{\partial P}{\partial S}(r-\delta)S+\frac{1}{2}\frac{\partial^2 P}{\partial S^2}\sigma^2 S^2 - rP = 0$

b. With final payoff/condition:
P(S,T)=g(S)=max(K-S,0).

c. The boundary conditions are:
$P(0,t)=K\,e^{-r(T-t)}$ and $P(\infty,t)=0$.

2. Since an American Put can be exercised at any time prior to maturity, non-arbitrage implies

$p(S,t) \geq \max(K-S,0)$ <u>at any date</u> $0 \leq t \leq T$.

a. At each time t prior to expiration, there exists a value S_t^{EP} for the stock price such that:

- If $S_t \leq S_t^{EP}$, early exercise is optimal and we have $p(S_t,t) = \max(K-S_t,0)$.

- If $S_t > S_t^{EP}$, immediate exercise is not optimal, and we have $p(S_t,t) > \max(K-S_t,0)$.

- S_t^{EP} is thus defined as the largest value of the stock price at t for which $p(S_t,t) = \max(K-S_t,0)$.

b. For an American call option:

$c(S_t,t) \geq \max(S_t-K,0)$, and there exists a value of the stock S_t^{EC} (when $\delta>0$) such that:

- If $S_t > S_t^{EC}$, early exercise is optimal, and we have $c(S_t,t) = \max(S_t-K,0)$

- If $S_t \leq S_t^{EC}$, early exercise is not optimal, and we have $c(S_t,t) > \max(S_t-K,0)$.

c. The exercise frontiers/boundaries S_t^{EP} and S_t^{EC} are unknown a priori.

3. Smooth Pasting Condition

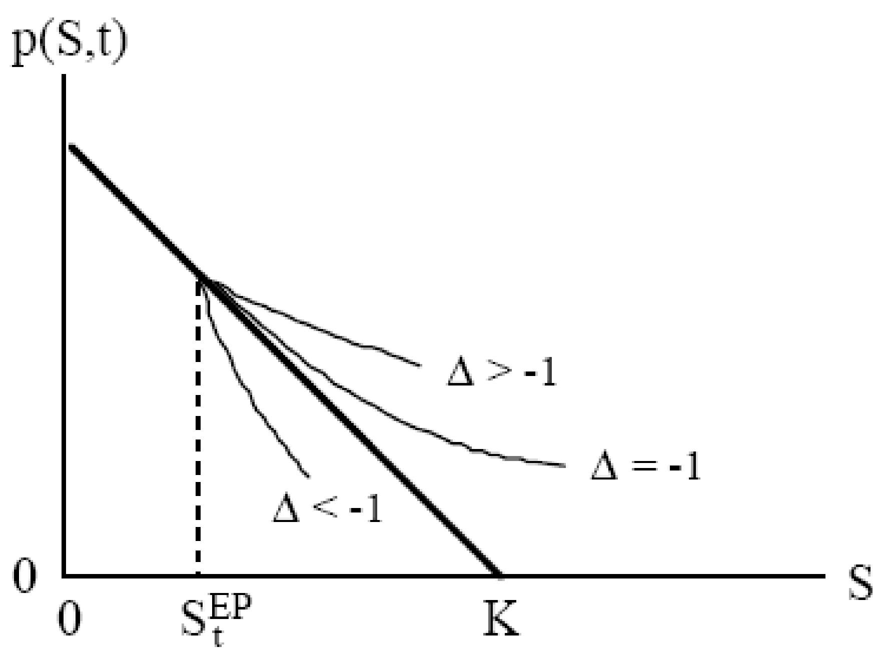
When S hits the barrier S^{EP}, the option price function must have a continuous derivative with respect to S, i.e. the option's delta must be continuous.

a. If $S \leq S^{EP}$, then p(S,t)=K-S, so $\lim_{S \xrightarrow{<} S^{EP}} \frac{\partial p}{\partial S} = -1$ (This is the limit to the left of S^{EP}).

b. Now consider the limit to the right: $\lim_{S \xrightarrow{>} S^{EP}} \frac{\partial p}{\partial S}$

The three possibilities are: <-1, >-1 or =-1.

The following graph illustrates the situation:



c. If $\Delta<-1$, then what this means is that dP/dS is less than -1, and so a change from S to S+dS implies a move (by the put price) below the intrinsic value line, which we know is impossible by non-arbitrage (since you could buy the American put option at a price below the intrinsic value line, buy the stock at S, exercise the put immediately which means you get to sell the stock at K, and collect the difference of K-S which would be more than the put cost you and thus an instantaneous arbitrage profit).

d. If $\Delta>-1$, then there exists an arbitrage opportunity. Consider purchasing one put option and one share of the stock.

- If $dS\leq0$, the moves takes us to the left of the exercise boundary, and we thus have:
  $p(S)=K-S$, $p(S+dS)=K-(S+dS)$
  so $dp= p(S+dS)-p(S) = -dS$ (>0 since $dS<0$).
  and so $d\Pi = dp(>0) + dS(<0) =0$.
  (Stated more informally, the put moves in the "Northwest" direction at a 45 degree angle by $|dS|$ and the stock loses $|dS|$).

- If $dS>0$, the small move dS creates a profit opportunity since $dS>0$ and $dp<0$ but in absolute value dp is smaller than dS. We thus have:
  $d\Pi=dS$ (positive) + dp (negative) >0

What is the order of magnitude of $d\Pi$?

• The order of magnitude of the profit is $\sqrt{dt}$ and not dt. The reason is that for dS=μSdt+σSdZ, we have:

$$
\begin{aligned}
d\Pi &= dS + dp \\
&= \underbrace{\left\{\text{drift of } dS + \text{drift of } dp\right\}dt}_{o(dt)} + \left\{\sigma S + \frac{\partial p}{\partial S}\sigma S\right\}dZ
\end{aligned}
$$

(by Ito's lemma)

• Therefore the expected profit is:

$$
\begin{aligned}
E\left[d\Pi \mid dS > 0\right] &= o(dt) + \left\{1 + \frac{\partial p}{\partial S}\right\}\sigma S\, o\left(\sqrt{dt}\right) \\
&= o\left(\sqrt{dt}\right) \text{ and positive if } \frac{\partial p}{\partial S} > -1
\end{aligned}
$$

• But we know that there cannot be a riskless profit opportunity yielding a return of order $\sqrt{dt}$ (The return should be the much smaller quantity rΠdt).

•Therefore the situation where Δ>-1 is impossible.

e. Thus we must have Δ=-1 at $S=S^{EP}$.

f. The condition Δ=-1 at $S=S^{EP}$ is known as the "high contact condition" or the "smooth pasting condition". <u>Reference</u>: Merton (1973), Bell Journal.

4. The American Put Option must thus satisfy for $0 \le t < T$ and $0 \le S \le \infty$:

a. For $S \le S_t^{EP}$, $p(S,t) = \max(K-S,0)$

b. The free boundary S_t^{EP} has the following characteristics:

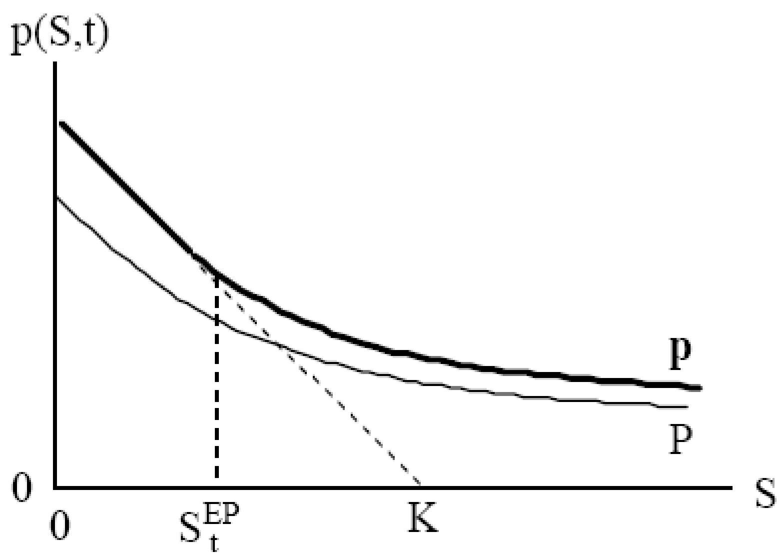
$$p(S_t^{EP},t) = K - S_t^{EP} \quad \text{and} \quad \frac{\partial p}{\partial S}\left(S_t^{EP},t\right) = -1.$$

c. For $S > S_t^{EP}$, $p(S,t)$ solves:

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial S}\left(r-\delta\right)S + \frac{1}{2}\frac{\partial^2 p}{\partial S^2}\sigma^2 S^2 - r\,p = 0$$

with $p(S,t) > \max(K-S,0)$ except for when t goes to T, where we then have $p(S,T)=\max(K-S,0)$.

d. Finally, the fixed boundary condition is: $p\left(\infty,t\right)=0$.



5. <u>Similarly, the price of an American call option satisfies for $0 \le t < T$ and $0 \le S \le \infty$</u>:

a. For $S \ge S_t^{EC}$, c(S,t) = max(S-K,0): this corresponds to optimal early exercise.

b. The free boundary S_t^{EC} is characterized by:

$$c(S_t^{EC},t) = S_t^{EC}\text{-}K \quad \text{and} \quad \frac{\partial c}{\partial S}\left(S_t^{EC},t\right)=1.$$

c. For $S < S_t^{EC}$: c(S,t) solves:

$$\frac{\partial c}{\partial t}+\frac{\partial c}{\partial S}(r-\delta)S+\frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 - r\,c = 0$$

And c(S,t) > max(S-K,0) except when t goes to the limit T, where c(S,T)=max(S-K,0).

d. The fixed boundary condition is: c(0,t)=0.

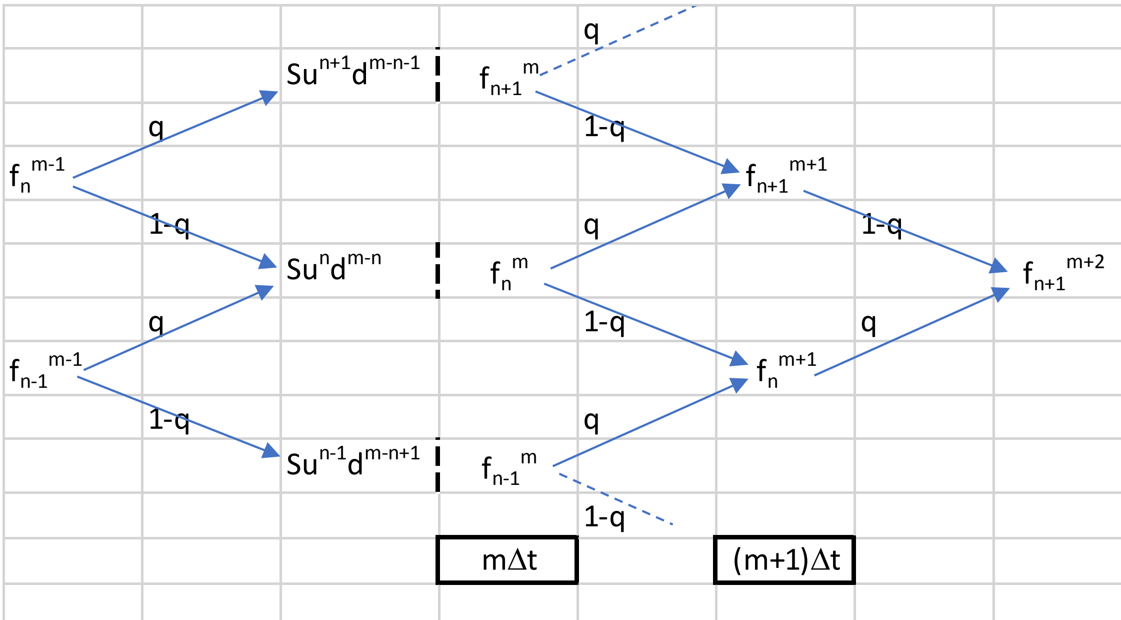
6. <u>Free boundary problems do not have known closed-form solutions</u>

a. The main problem is that the location of the free boundary is unknown and must be determined as part of the solution.

b. The Black-Scholes assumptions that help us in the case of fixed boundary problems are of no help here.

C. Modifying the Cox-Ross-Rubinstein Tree

1. Recall the CRR Tree:



a. Where f_n^m = value of the derivative at node (n,m), indicating time mΔt and stock went up n times.

b. Setting u, d, and q to match the two moments of the (risk-neutral) diffusion

$$dS = (r-\delta)Sdt + \sigma SdZ_t^Q$$

allows us to use risk-neutral valuation backward:

- In the example of a European Put:

$$f_n^m = \frac{1}{1+r\Delta t}\left\{q f_{n+1}^{m+1} + (1-q) f_n^{m+1}\right\}$$

- However, in the case of an American put, we must check at each node whether early exercise is optimal, thus the value of the derivative at each node is:

$$f_n^m = \max\left(\frac{1}{1+r\Delta t}\left\{q\,f_{n+1}^{m+1} + (1-q)\,f_n^{m+1}\right\},\; K - Su^n d^{m-n}\right)$$

• The procedure is otherwise the same as with a European option, with the final condition being:

$$f_n^M = \max\left(0, K - Su^n d^{M-n}\right),\ n=0,1,...,M.$$

2. Notice that the value of the derivative at time $m\Delta t$ accounts for both the possibility of early exercise at time $m\Delta t$ and the possibility of exercise at time $(m+1)\Delta t$, $(m+2)\Delta t$, etc… The premium specific to American options is thus built recursively into the prices of the derivative.

D. <u>Modifying Finite-Difference Methods</u>

Finite-difference grids can also accommodate the pricing of American options by adding a condition specifying that for all n and m, the value of the derivative f must be above its intrinsic value.

It is therefore a very straightforward procedure.