Existence problems for the p-Laplacian

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Abstract

We consider a number of boundary value problems involving the p-Laplacian. The model case is $-\Delta_p u = V|u|^{p-2}u$ for $u \in W_0^{1,p}(D)$ with D a bounded domain in $\mathbf{R}^n$. We derive necessary conditions for the existence of nontrivial solutions. These conditions usually involve a lower bound for a product of powers of the norm of V, the measure of D, and a sharp Sobolev constant. In most cases, these inequalities are best possible. Applications to nonlinear eigenvalue problems are also discussed.

1. Introduction

Let D be an open bounded region in $\mathbf{R}^n$, with $n \geq 1$. Define the p-Laplacian by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{1.1}$$

for $1 < p < \infty$. Apart from its intrinsic interest, the p-Laplacian arises in the study of non-Newtonian fluid mechanics both for $p \geq 2$ (dilatant fluids) and $1 < p < 2$ (pseudoplastic fluids), see [AS]. It also arises in the study of of quasiconformal mappings [Ho] and other topics in geometry [U]. The one dimensional case also arises in models of turbulent flow of gas in porous media [OR]. In this work, we consider equations such as

$$-\Delta_p u = V|u|^{p-2}u, \ u \in W_0^{1,p}(D), \tag{1.2}$$

where V is assumed to be real-valued and integrable, and u is assumed to be complex-valued unless stated otherwise. We assume nothing about the boundary of D. This paper generalizes previous work by the authors [DEHL], (also see [DH1],[DH2]), where the case $p = 2$ is considered. Of course for $p = 2$, Δ_p is the well-known linear Laplace operator.

Assuming a non-trivial solution to (1.2), we prove inequalities of the form

$$K\|V\|_r \geq 1.$$

To state a typical result, we need more notation. Suppose $1 < p < n$ and $q < \frac{np}{n-p}$. Let $K = K_{q,p} = K(q,p,n,D)$ be the Sobolev constant of the embedding $W_0^{1,p}(D) \to L^q(D)$. That is,

$$K_{q,p} = \sup_{u\neq 0} \frac{||u||_q}{||\nabla u||_p}. \tag{1.3}$$

By Lemma 5.2, equality is attained in (1.3) by a nonnegative *extremal* function u_*. We have similar results (see Lemma 5.1) and adopt similar notation when $p > n$ and $q \leq \infty$. Our first such result is:

Theorem 1.1 *Suppose $1 < p < n$ and $\frac{n}{p} < r \leq \infty$. Let q satisfy $1/r + p/q = 1$, so that $q < \frac{np}{n-p}$. Suppose $V \in L^r(D)$, and $u \in W_0^{1,p}(D)$ is a nontrivial solution of (1.2). Then*

$$K_{q,p}^p ||V_+||_r \geq 1. \tag{1.4}$$

Let $u_* \geq 0$ be an extremal for (1.3) given by Lemma 5.2. Then (1.2) holds with $u = u_*$, and equality in (1.4) is attained when

$$V(x) = c_* u_*^{q-p}(x), \quad \textit{where} \quad c_* = \|u_*\|_q^{-q}.$$

The proof of Theorem 1.1 is presented in Section 2, where we also consider the case $p > n$ with $r \geq 1$, and the case $p < n$ with $r \leq n/p$. We consider the case $p = n > 1$ in Section 3. The variations of Theorem 1.1 presented in the paper complement results in [OR] and [BGG] and references therein, where sufficient conditions for the existence of solutions to (1.2) are proved for $n = 1$.

A key ingredient here is the imbedding of $W_0^{1,p}(D)$ into various Banach spaces, based on inequalities of Sobolev or Moser-Trudinger. An interesting theme here is that (1.2) is the Cauchy-Euler equation for the extremals of these imbeddings. The associated functions V are extremals for our proclaimed lower bounds, such as (1.4). For a related discussion, see [H].

We can estimate the Sobolev constant $K(q,p,n,D)$ in terms of $|D|$. Let B be a ball with $|B| = 1$, and let $K_{q,p}^* = K(q,p,n,B)$. Standard arguments involving scaling and symmetrization, [LL], show that $K(q,p,n,D) \leq |D|^{1/q-1/p+1/n} K_{q,p}^*$. Inserting this into (1.4), we have

$$|D|^{p/q-1+p/n} (K_{q,p}^*)^p ||V_+||_{(q/p)'} \geq 1. \tag{1.5}$$

This is a minimal support result, which is a special type of unique continuation result, see [DEHL]. That is, if $|D|$ is too small to satisfy (1.5), then any solution to (1.2) must vanish on D. Similar remarks apply to many other results in this paper.

Equation (1.2) can be viewed as a generalization of the non-linear eigenvalue equation: $-\Delta_p u = E|u|^{p-2}u$, $u \in W_0^{1,p}(D)$, where $E \in \mathbf{R}$. Such eigenvalue problems have been studied by a number of authors, see [Le] and references therein. A further generalization follows easily by applying Theorem 1.1 to the function $V + E$:

Corollary 1.2 *Let p, q, r be as in Theorem 1.1. Let $V \in L^r(D)$. Let $u \in W_0^{1,p}(D)$ be a nontrivial solution of*

$$-\Delta_p u - V|u|^{p-2}u = E|u|^{p-2}u, \tag{1.6}$$

where $E \leq 0$ is a constant. Then $K^p_{q,p}||(V+E)_+||_r \geq 1$.

This result can by viewed as a lower bound on the eigenvalues of (1.6). Sufficient conditions for solvability of (1.6) can be found in [HR]. The case of Neumann boundary conditions with $V \geq 0$, along with application to degenerate parabolic equations, is studied in [B].

The following corollary is basically a rephrasing of Theorem 1.1:

Corollary 1.3 *Let p, q, r be as in Theorem 1.1. Let $\mathcal{G} = \{V \in L^r(D) : V(x) \geq 0, \|V\|_r = 1\}$. Then among all pairs $(E, V) \in \mathbf{R}^+ \times \mathcal{G}$ for which there exists non-trivial $u \in W^{1,p}_0(D)$ such that*

$$-\Delta_p u = EV|u|^{p-2}u, \tag{1.7}$$

we have

$$E \geq \frac{1}{K^p_{pq}}.$$

Equality is attained when $u = u_*$, as defined in Lemma 5.2, with $V = u_*^{q-p}/\|u_*^{q-p}\|_r$.

Note that $1/K^p_{pq}$ is the smallest eigenvalue of (1.7), and u_* the corresponding eigenfunction. Corollary 1.3 can be compared with the following result in [CEP]. Fix $V_0 \geq 0$ with $V_0 \in L^\infty$ and $||V_0||_r = 1$. Let $\mathcal{G}_1$ be the set of all measurable rearrangements of V_0. Then among all pairs $(E, V) \in \mathbf{R}^+ \times \mathcal{G}_1$ for which there exists a positive $u \in W^{1,p}_0(D)$ such that (1.7) holds, there exists V_1 that minimizes E. Furthermore, letting u_1 be the corresponding positive, normalized eigenfunction, there exists an increasing function ϕ such that $V_1 = \phi(u_1)$. A formula for ϕ is not given, and seems to be difficult to deduce from the methods in [CEP]. It would be interesting to see how V_1, ϕ relate to the pair $V, z \to z^{q-p}$ appearing in Corollary 1.3. We remark that analogues of Corollaries 1.2 and 1.3 hold for the variants of Theorem 1.1 that will follow.

Our paper is organized as follows. In Section 2, we present lower bounds on $||V||_r$ based on the inequalities of Sobolev and present examples to demonstrate sharpness. In Section 3, we present lower bounds on an Orlicz norm of V, based on the Moser-Trudinger inequality. In Section 4, we study two generalizations of (1.2): the equation $-\Delta_p u = V(x)|u|^\beta u$ with $\beta \neq p-2$, and the equation $-\Delta_p u = V(x)f(x, u, \nabla u)$. Finally, we have collected some technical lemmas, perhaps not entirely new, in an appendix.

2. L^r lower bounds.

In this section we assume that the potential V belongs to some Lebesgue space $L^r(D)$, with $r \geq 1$, and then show that $||V||_r$ must be large.

2.1 The basic theorems

We assume $u \in W^{1,p}_0(D)$ and that (1.2) holds in the distribution sense. That is, for every $\psi \in C^\infty_0(D)$,

$$\int_D |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\psi(x)\, dx = \int_D V(x)|u(x)|^{p-2}u(x)\psi(x)\, dx.$$

Define $p^* = \frac{np}{n-p}$ for $p < n$, and $p^* = \infty$ for $p \geq n$. Recall that if $1 < p < n$, then $u \in L^q(D)$ for $1 \leq q \leq p^*$. If $p > n$, then $u \in C^0(D) \cap L^q(D)$ for $1 \leq q \leq \infty$. Unless stated otherwise, we assume $p \leq q$ and $r = (q/p)'$ is the Holder conjugate of q/p, meaning $1/r + p/q = 1$, as in Thm. 1.1.

In Theorem 2.1, we consider cases where (1.4) and (1.3) have no extremal, for example, when $p < n$ and $q = p^* < \infty$. Note that in effect $r = (p^*/p)' = n/p$. In this case, the Sobolev constant does not depend on D and can be computed explicitly [T]. For brevity in proofs, we may abbreviate it, letting $K = K_{q,p}$. For $p > n$, extremals do exist for the Sobolev constant $K_{\infty,p}$, but they don't solve (1.2) with $V \in L^1(D)$, see Section 2.2.

Theorem 2.1 *Let* $1 < p < n$ *and let* $u \in W_0^{1,p}(D)$ *be a nontrivial solution of (1.2), with* $V \in L^{n/p}(D)$. *Then*

$$K^p_{p^*,p} ||V_+||_{n/p} > 1. \tag{2.1}$$

If $p > n$ *and* $V \in L^1(D)$, *then*

$$K^p_{\infty,p} ||V_+||_1 > 1. \tag{2.2}$$

The constant 1 is sharp in both (2.1) and (2.2) when D *is a ball.*

If desired, one can replace $||V_+||_{n/p}$ in (2.1) or $||V||_1$ in (2.2) by $||V||_{n/p}$, $||V||_1$ respectively. In that case the constant 1 is still sharp, with the same proof.

A maximal principle formulated in ([GT], Theorem 10.10) implies an inequality of the form $c\|V_+\|_{n/p} > 1$, similar to (2.1). But it is not clear whether the constant c is sharp.

The following extension of Theorem 1.1 handles the cases where $V \in L^r$ with $r > n/p$ for $p < n$, and $r > 1$ for $p > n$. In both cases, the Sobolev embedding is compact, and we have extremals for (1.3) satisfying (1.2) with V in a Lebesque space.

Theorem 2.2 *Suppose* $\max(1, \frac{n}{p}) < r \leq \infty$. *Suppose* $V \in L^r(D)$, *and* $u \in W_0^{1,p}(D)$ *is a nontrivial solution of (1.2). Then*

$$K^p_{q,p} ||V_+||_r \geq 1. \tag{2.3}$$

Let $u_* \geq 0$ *be an extremal for (1.3) given by Lemmas 5.1 or 5.2. Then (1.2) holds with* $u = u_*$, *and equality in (2.3) is attained, when*

$$V(x) = c_* u_*^{q-p}(x), \quad \textit{where} \quad c_* = \frac{1}{||u_*||_q^q}. \tag{2.4}$$

We have the following simple consequence of Theorems 2.1 and 2.2, which includes Corollary 1.2 as a special case.

Corollary 2.3 *Let* $u \in W_0^{1,p}(D)$ *be a nontrivial solution of*

$$-\Delta_p u - V|u|^{p-2}u = E|u|^{p-2}u.$$

where $E \leq 0$ *is a constant.*

(a) If $p < n$ *and* $V \in L^{n/p}(D)$*, then* $K^p_{p^*,p}||(V+E)_+||_{n/p} > 1$.
(b) If $p > n$ *and* $V \in L^1(D)$*, then* $K^p_{\infty,p}||(V+E)_+||_1 > 1$.
(c) If $\max(1, \frac{n}{p}) < r \leq \infty$ *and* $V \in L^r(D)$*, then* $K^p_{q,p}||(V+E)_+||_r \geq 1$.

The obvious analogue of Corollary 1.3 also holds.

Proof of Thm 2.2: Since $\max(1, \frac{n}{p}) < r \leq \infty$ we have $q \leq p^*$, which allows Sobolev's inequality (2.5) below. Also, using Green's identity (see Lemma 5.4), and Holder's inequality based on $1/r + p/q = 1$,

$$\begin{aligned} \|u\|_q^p &\leq K^p \int_D |\nabla u|^p dx & (2.5) \\ &= K^p \int_D V|u|^p dx \\ &\leq K^p \int_D V_+|u|^p dx & (2.6) \\ &\leq K^p \|V_+\|_r \|u\|_q^p. & (2.7) \end{aligned}$$

Since $\|u\|_q > 0$, we have $K^p\|V_+\|_r \geq 1$, and (2.3) holds. This type of proof, which also appears in [DEHL] and later in this paper, will be referred to informally as a *minimal support sequence.*

We now show that equality can be attained in (2.3). By Lemmas 5.1, 5.2 and 5.3, there is a $u_* \geq 0$ for which (2.5) holds with equality, with $\|\nabla u_*\|_p = 1$ and

$$-\Delta_p u_* = \frac{u_*^{q-p}}{\|u_*\|_q^q} u_*^{p-1}. \tag{2.8}$$

Setting $V = u_*^{q-p}/\|u_*\|_q^q$, we also have equality in (2.6). Equality occurs in (2.7) because $(|u_*|^p)^{\frac{q}{p}-1} = |u_*|^{q-p} = cV_+$, see for example [LL], p.45. □

For more insight into why the extremal for Sobolev's inequality generates the extremal for (2.3), the reader is referred to [H].

Remark: the method of proof of Theorem 2.2 can easily be applied to study the equation $-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = V|u|^{p-2}u$, where a is a positive function with $a, 1/a \in L^\infty$. Other results in this paper can also be generalized in this way.

Proof of Thm 2.1: We begin with the case $p < n$. The proof of Thm 2.2 shows that $K^p\|V_+\|_r \geq 1$ when $r = (q/p)' = (p^*/p)' = n/p$. It is well-known [T] that when $q = p^*$, and u is nontrivial, equality cannot occur in (2.5) for any bounded domain D. So, $K^p\|V_+\|_{n/p} > 1$, as desired.

To show that the constant 1 is sharp in (2.1), we will set $D = B_{\hat{R}}(0)$ and construct u, V on D so that $K^p||V||_{n/p} \to 1$ as $\hat{R} \to \infty$. Since $1 < K^p||V_+||_{n/p} \leq K^p||V||_{n/p}$, this implies $K^p||V_+||_{n/p}$ can be arbitrarily close to 1. Let $\rho = |x|$. For radial u,

$$\Delta_p u(\rho) = |u_\rho|^{p-2}((p-1)u_{\rho\rho} + \frac{n-1}{\rho}u_\rho) = (p-1)|u_\rho|^{p-2}(u_{\rho\rho} + \frac{s-1}{\rho}u_\rho), \tag{2.9}$$

for $s = \frac{n-1}{p-1} + 1$. The extremal for the Sobolev embedding $W_0^{1,p}(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ for $1 < p < n$ with critical index $q = pn/(n-p)$ is given by

$$v(\rho) = (1 + \rho^{p'})^{(p-n)/p}, \tag{2.10}$$

where $p' = p/(p-1)$, see [T]. Define V_v by $-\Delta_p v = V_v v^{p-1}$. Applying the minimal support sequence to v, V_v with $r = n/p$, and with $D = \mathbf{R}^n$ (temporarily), we get $K^p||V_v||_r = 1$.

Below, let C be a positive constant whose value may change at each step, and let R be a constant that eventually will approach infinity, so we can assume without loss of generality that $R > C$. The other constants below may depend on R, but K does not, as it is independent of dilation in this critical case. Let $\hat{R} > R + 1$ to be specified later. Set

$$u = \begin{cases} v, & 0 \le \rho < R, \\ a - b\rho, & R \le \rho < R+1, \\ c\rho^{2-s} + d, & R+1 \le \rho \le \hat{R}. \end{cases}$$

To make u and u_ρ continuous at $\rho = R$ and at $\rho = R+1$, let

$$\begin{aligned} a - bR &= v(R), \\ b &= \frac{(n-p)}{(p-1)} R^{p'-1}(1 + R^{p'})^{-n/p}, \\ c &= (R+1)^{s-1} R^{p'-1}(1 + R^{p'})^{-n/p}, \\ d &= (1 - \frac{(n-p)}{(p-1)} R^{p'-1})(1 + R^{p'})^{-n/p}. \end{aligned}$$

Note that by (2.10), $b = |v_\rho(R)| \le Cv(R)/R$ for some C independent of R. Assuming $R > C$,

$$u(R+1) = a - b(R+1) = v(R) - b \ge v(R)(1 - C/R) > 0 \tag{2.11}$$

Since $\lim_{\rho\to\infty} c\rho^{2-s} + d = d < 0$ and u is continuous, there is some $\hat{R} > R+1$ so that $u(\hat{R}) = 0$. Set $D = B_{\hat{R}}(0)$, and note that $u \in W_0^{1,p}(D)$ is radial and nonnegative. Define V_u by $-\Delta_p u = V_u u^{p-1}$. For $R \le \rho < R+1$, we get $|V_u| = |\frac{(n-1)b^{p-1}}{\rho(a-b\rho)^{p-1}}| \le CR^{-p}$. For $R+1 \le \rho$, we have

$$\Delta_p(\rho^{2-s}) = 0, \tag{2.12}$$

hence $V_u = 0$. So,

$$\int_D |V_u|^{n/p} \le C \int_{R\le\rho\le R+1} R^{-n} + \int_{B_R(0)} |V_v|^{n/p}.$$

As $R \to \infty$, the first integral is bounded by CR^{-1} which converges to 0. The second integral converges to $||V_v||_{n/p}^{n/p} = K^{-n}$, so $K^p\|V_u\|_{L^{n/p}(D)} \to 1$, as desired. This proves sharpness of (1.2) when $p < n$.

We now address the case $p > n$. The proof of Thm 2.2 applies, with $q = p^* = \infty$ and $r = 1$, and hence $K^p\|V_+\|_1 \ge 1$. We will show in the next subsection, in Thm 2.4, that the last inequality is strict, i.e. $K^p\|V_+\|_1 > 1$, and that the constant 1 cannot be improved, completing the proof. □

2.2 Inequalities for bounded solutions with $V \in L^1$

If $p > n$ then $u \in L^\infty$ and we may consider $V \in L^1(D)$ in Theorem 2.1. The lower bound for $||V_+||_1$ still holds, but the usual Euler-Lagrange equation (5.2) does not, raising interesting new questions about sharpness and extremals. We prove an analogue of Theorem 2.1 replacing $L^1(D)$ with the space M of signed measures V on D, (see e.g. [Ru] for the definition and properties of signed measures) with norm $\|V\|_M = |V|(D) < \infty$. In the special case where $V \in L^1(D)$, we have $\|V\|_M = |V|(D) = \int_D |V(x)|dx = \|V\|_1$. We also recall $V_+ = \frac{1}{2}(V + |V|)$. Assume (1.2) holds for $V \in M$, i.e.

$$\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\phi =< V, |u|^{p-2}u\phi >, \ \forall \phi \in C_0^\infty(D).$$

Here $< *, * >$ denotes the pairing of signed measures with continuous functions. Let $\phi_j \in C_0^\infty(D)$ converge in $W^{1,p}(D)$, and hence in $C^0(D)$, to $\overline{u}$. It then follows that

$$\int_D |\nabla u|^p =< V, |u|^p > . \tag{2.13}$$

Theorem 2.4 *Assume $u \in W_0^{1,p}(D)$ is a nontrivial solution of (1.2), with $V \in M$. Then*

$$K^p_{\infty,p}||V_+||_M \geq 1. \tag{2.14}$$

Equality is attained when $V = K^p_{\infty,p}\ \delta_z(x)$, where δ_z is a Dirac mass at some point $z \in D$. When $V \in L^1(D)$, equality is not possible, but the constant 1 cannot be replaced by any larger constant.

Proving the last assertion of this theorem will complete the proof of Theorem 2.1.

Proof: Let $K = K_{\infty,p}$. Using (2.13), we have the minimal support sequence

$$\begin{aligned} \|u\|_\infty^p &\leq K^p\|\nabla u\|_p^p \\ &= K^p < V, |u|^p > \\ &\leq K^p < V_+, |u|^p > \\ &\leq K^p\|V_+\|_M\|u^p\|_\infty. \end{aligned} \tag{2.15}$$

The inequality (2.14) follows immediately. We now demonstrate extremals for this inequality. By Lemma 5.1, there is a non-negative extremal $u_* \in W_0^{1,p}(D)$ for the Sobolev inequality used in (2.15), that is

$$||u_*||_\infty = K||\nabla u_*||_p \ . \tag{2.16}$$

This u_* represents a scalar multiple of the extremal in Lemma 5.1. That is, we do not assume $||\nabla u_*||_p = 1$. Since u_* is continuous, it attains its maximum value at some point $z \in D$. So, (2.16) can be rewritten

$$K = \frac{|u_*(z)|}{||\nabla u_*||_p}.$$

where u_* maximizes the right-hand side among all $u \in W_0^{1,p}(D)$. We now apply the Euler-Lagrange method to this. Let $\phi \in C_0^\infty(D)$, and let $u_\epsilon(x) = u_*(x) + \epsilon\phi(x)$. Then

$$0 = \frac{d}{d\epsilon}\left(\frac{|u_\epsilon(z)|}{||\nabla u_\epsilon||_p}\right)|_{\epsilon=0}$$

gives

$$0 = \phi(z) - u_*(z)\|\nabla u_*\|_p^{-p}\int_D |\nabla u_*|^{p-2}\nabla u_* \cdot \nabla\phi dx.$$

Normalizing, we can assume $u_*(z) = 1$, so $1 = K\|\nabla u_*\|_p$, and the above shows u_* is a weak solution to

$$-\Delta_p u = K^{-p}\delta_z, \tag{2.17}$$

and it satisfies (1.2) with $V_* = K^{-p}\delta_z$. Since $||\delta_z||_M = 1$, we have $K^p||V_*||_M = 1$, and hence V_* is an extremal.

Now suppose (2.14) was an equality for a $V \in L^1(D)$ with corresponding solution u to (1.2). Then the argument leading to (2.15) would imply $\|u\|_\infty^p = K^p\|\nabla u\|_p^p$, so u would be an extremal for the Sobolev inequality. Hence, by the argument above, $-\Delta_p u = K^{-p}\delta_z|u|^{p-2}u = V|u|^{p-2}u$ as distributions, for some $z \in D$. This implies $V|u|^{p-2}u = 0$ a.e., so that $\Delta_p u = 0$ as a distribution. But then $u \equiv 0$, a contradiction.

Next, to show that the constant 1 in (2.14) is sharp for $V \in L^1$, we construct examples on $D = B_1(0)$ such that $K^p||V||_1 \to 1$. Define u_* as a Sobolev extremal as above. By standard symmetrization, we may assume u_* is radially non-increasing, so that $z = 0$, and by (2.17) $-\Delta_p u_* \equiv 0$ away from 0. Using (2.9) and solving an ordinary differential equation, we get $u_*(\rho) = 1 - \rho^{\frac{p-n}{p-1}}$.

We approximate u_* by a function u such that $V_u = \frac{\Delta_p u}{u^{p-1}} \in L^1$. Let $\epsilon > 0$ and let $u = u_*$ outside $B_\epsilon(0)$. On $B_\epsilon(0)$, define $u(\rho) = a - b\rho^2$ such that $u'(\epsilon)$ exists. Note $\Delta_p u \in L^1(B_1(0))$, so $V_u \in L^1$ too. Also, $\Delta_p u \le 0$ there, because $p > n$, $b > 0$, and $u'' + (n-1)u'/((p-1)\rho) = -2b[1 + (n-1)/(p-1)] < 0$. So $V_u \ge 0$, and

$$||V_u||_1 = \int_D V_u\ dx \le u_*^{1-p}(\epsilon)\int_D V_u u^{p-1}\ dx = -u_*^{1-p}(\epsilon)\int_D \Delta_p u\ dx.$$

Let $v \in C_0^\infty(D)$ with $v \equiv 1$ on $B_\epsilon(0)$. Then

$$\int_D \Delta_p u\ dx = \int_D v\ \Delta_p u\ dx = -\int_D |\nabla u|^{p-2}\nabla u \ \cdot \nabla v\ dx = -\int_D |\nabla u_*|^{p-2}\nabla u_* \ \cdot \nabla v\ dx.$$

By (2.17) and the definition of $\Delta_p u_*$, we get $\int_D \Delta_p u\ dx = -K^{-p}$. As $\epsilon \to 0$, we have $u_*(\epsilon) \to 1$, so that $||V_u||_1 \to K^{-p}$, as desired.

2.3 On the failure of other norms on V

The proofs of Thms. 2.1 and 2.2 depend on the Sobolev and Holder inequalities, which impose the restriction $r \ge n/p$ (when $p < n$). In this section, we prove that if $r < n/p$, $||V||_r$ can be arbitrarily small.

Theorem 2.5 *Let $p < n$ and $D = B_1(0) \subset \mathbf{R}^n$. For every $1 \le r < \frac{n}{p}$, and every $\delta > 0$, there is a potential V_δ, with a nontrivial solution $u \in W_0^{1,p}(D)$ of (1.2), with $\|V_\delta\|_r < \delta$.*

Proof. We will specify $\epsilon = \epsilon(\delta) \in (0, 1/2)$ later. With the usual convention that $u(x) = u(|x|) = u(\rho)$, define

$$u(\rho) = \begin{cases} a - b\rho^{\frac{p}{p-1}}, & 0 \le \rho < \epsilon. \\ \rho^{2-s} - 1, & \epsilon \le \rho \le 1, \end{cases}$$

where $s = \frac{n-1}{p-1} + 1$, and a, b are chosen to make $u \in C_0^1(D)$. So, $b = \frac{n-p}{p}\epsilon^{-\frac{n}{p-1}}$ and $a = \frac{n}{p}\epsilon^{2-s} - 1$. Since u is radial, by (2.9) we have

$$\Delta_p u(\rho) = (p-1)|u_\rho|^{p-2}(u_{\rho\rho} + \frac{s-1}{\rho}u_\rho).$$

On $\epsilon \le \rho \le 1$, by (2.12) we have $\Delta_p u(\rho) = 0$ and $V_\delta \equiv 0$. Let C denote positive constants that vary from line to line. For $0 \le \rho < \epsilon$, $u_\rho(\rho) = -\frac{p}{p-1}b\rho^{\frac{1}{p-1}}$, $u_{\rho\rho}(\rho) = -\frac{p}{(p-1)^2}b\rho^{\frac{1}{p-1}-1}$ and $-\Delta_p u(\rho) = C\epsilon^{-n}$; hence $|V_\delta| = \frac{C\epsilon^{-n}}{|u|^{p-1}} \le C\epsilon^{-p}$. So, $\|V_\delta\|_r^r \le C\epsilon^{n-rp} < \delta$, for small enough ϵ. $\square$

3. The case $p = n$: Orlicz lower bounds.

3.1 The critical case $p = n \ge 2$: $V \in L\log^{n-1} L(D)$

When $p = n$, the proof of Theorem 2.2 does not extend to $r = \frac{n}{p} = 1$, because $W_0^{1,n}(D)$ does not embed into $L^\infty(D)$. In this section we assume instead that V is in the Orlicz space $L\log^{n-1} L(D)$, so that $\int_D |V|(\log^{n-1}(1 + |V|))dx$ is finite, and prove an analogue of Theorem 2.2 for that space. By Theorem 3.6 no such analogue holds for $V \in L\log^k L(D)$, with $0 \le k < n-1$, and therefore not for $V \in L^1$.

As a substitute for the Sobolev inequality we will use the Moser-Trudinger inequality (see [M]). Let $\alpha_n = (n^{n-1}\omega_n)^{1/n}$, where ω_n is the surface area of the unit sphere in $\mathbf{R^n}$. Suppose $u \in W_0^{1,n}(D)$ is real-valued. The inequality is

$$\int_D \exp((\alpha_n |u(x)|/\|\nabla u\|_n)^{\frac{n}{n-1}})\ dx \le C_n|D|.$$

Let $0 < \alpha < (\alpha_n)^n$ be a fixed constant. Define

$$M(t) = \int_0^{\alpha t} (e^{s^{\frac{1}{n-1}}} - 1)\ \ ds \tag{3.1}$$

and

$$N(s) = \int_0^{s/\alpha} \log^{n-1}(t+1)\ dt\ .$$

These are complementary Orlicz functions, see [KR]. This non-standard choice for M, N allows an explicit formula for $N(s)$, which is useful because N is used in the definition of the norm of V. Let $P_0(x) = 1$, and $P_m(x) = \sum_{k=0}^m \frac{(-1)^k m!}{(m-k)!}x^{m-k}$ for $m \ge 1$. Then, for $n \ge 2$,

$$N(s) = (1 + \frac{s}{\alpha})P_{n-1}(\log(1 + \frac{s}{\alpha})) + (-1)^n(n-1)!\ .$$

The Orlicz class $L^N(D)$ is the set of measurable u such that $\int_D N(|u|)dx < \infty$ with $L^M(D)$ defined similarly. Fix c, with $\alpha^{1/(n-1)} < c < (\alpha_n)^{n/(n-1)}$. By (3.1) there is a $C > 0$ such that both $M(t),\ M'(t) < Ce^{ct^{\frac{1}{n-1}}}$ for all $t > 0$. So the Moser-Trudinger inequality gives

$$\int_D M\left(\frac{|u(x)|^n}{\|\nabla u\|_n^n}\right) dx \le K_M|D|, \quad \forall u \in W_0^{1,n}(D), \tag{3.2}$$

where the optimal constant K_M depends on D, but is dilation-invariant and independent of u.

Remark: An example in [M] shows the (3.2) does not hold if $\alpha = (\alpha_n)^n$ for $n \ge 3$. The case $n = 2$ is discussed later in a remark below.

We define the norm of $V \in L\log^{n-1} L(D)$ by

$$\|V\|_N = \inf\left\{\lambda + \frac{\lambda}{K_M|D|}\int_D N\left(\frac{|V(x)|}{\lambda}\right) dx;\ \lambda > 0\right\} < \infty, \tag{3.3}$$

see [KR]. For fixed V, we set $F(\lambda) = \lambda \int_D N\left(\frac{V_+(x)}{\lambda}\right) dx$, so that $\|V_+\|_N = \inf\left\{\lambda + \frac{F(\lambda)}{K_M|D|}\right\}$. With the norm for L^M defined analogously, standard arguments show that the injection $W_0^{1,n}(D) \to L^M(D)$ is compact.

Lemma 3.1 *There exists a non-negative extremal u for (3.2). Furthermore, the Euler Lagrange equation for (3.2), with the normalization $||\nabla u||_n = 1$, is*

$$-\Delta_n u = V u^{n-1}$$

where

$$V(x) = \frac{M'(u^n(x))}{\int_D M'(u^n)u^n dx} \in L\log^{n-1} L(D). \tag{3.4}$$

Sketch of Proof: By the compactness of $W_0^{1,n}(D) \to L^M(D)$, there is an extremal for (3.2). Variational work similar to that done for Theorem 2.4 gives the equation in (3.4). That $V \in L\log^{n-1} L(D)$ follows from (3.1).

Our main result for the case $p = n$ is:

Theorem 3.2 *Assume that (1.2) has a nontrivial solution u for $V \in L\log^{n-1} L(D)$. Then*

$$K_M|D|\|V_+\|_N \ge 1, \tag{3.5}$$

where K_M is the optimal constant in (3.2). With $u = u_*$ and V as in Lemma 3.1, equality is attained in (3.5), and (1.2) holds.

Remark: The method of proof of Theorem 3.2 can be adapted to the Orlicz function

$$\tilde{M}(t) = e^{(\alpha_n^n t)^{1/(n-1)}} - \sum_{k=0}^{n-1} (\alpha_n^n t)^{k/(n-1)}/k!\ ,$$

for which, by [Li], (3.2) has extremals for $n \ge 2$. For $n = 2$, we observe this formula is (3.1) with $\alpha = (\alpha_2)^2$.

Theorem 3.2 follows immediately from the following:

Theorem 3.3 *Suppose that (1.2) has a nontrivial solution u with $V \in L\log^{n-1} L(D)$. Then, for every $\lambda > 0$,*

$$\lambda K_M |D| + F(\lambda) \geq 1. \tag{3.6}$$

Equality can be attained in (3.6) with u_* and V as in Lemma 3.1, and (1.2) holds.

Proof of Theorem 3.3. Fix u, V with $\|\nabla u\|_n = 1$. Let $U = |u(x)|^n$. For fixed $\lambda > 0$, set $v = \frac{V_+(x)}{\lambda}$. It is well known for any Orlicz pair (M, N) that Young's inequality gives

$$Uv \leq M(U) + N(v) \tag{3.7}$$

with equality if and only if $v = M'(U)$. By Green's identity (Lemma 5.4), the definition of U and (3.7),

$$\begin{aligned} 1 = \|\nabla u\|_n^n &= \int_D |u|^n V \, dx \\ &\leq \int_D U V_+ dx \\ &\leq \lambda \int_D M(U) dx + F(\lambda). \end{aligned}$$

By (3.2), $\displaystyle\int_D M(U) dx \leq K_M |D|$ and (3.6) follows.

Let u_* and V be as in Lemma 3.1, so that $-\Delta_n u_* = V u_*^{n-1}$, where $V = \omega^{-1} M'(u_*^n) \geq 0$, where

$$\omega = \int_D M'(u_*^n) u_*^n dx.$$

Let $U = U_* = u_*^n$, so $\int_D M(U_*) dx = K_M |D|$ (see (3.2)). Setting $\lambda = \omega^{-1}$, we have $v = V_+/\lambda = M'(U_*)$, so equality holds in (3.7). From the definitions of V, U_* and ω, we have $\int_D |U_* V| dx = 1$. Then, integrating (3.7),

$$1 = \lambda \int_D M(U_*) \; dx + F(\lambda) = \lambda K_M |D| + F(\lambda).$$

Thus for these choices of u, V and λ, (3.6) is an equality, and also (1.2) holds. $\square$.

Corollary 3.4 *Let $u \in W_0^{1,n}(D)$, and $V \in L\log^{n-1} L(D)$. If*

$$-\Delta_n u - V|u|^{n-2} u = E|u|^{n-2} u,$$

with $E \leq 0$, then

$$K_M |D| \; \|(V + E)_+\|_N \geq 1.$$

Corollary 3.5 *Let M, N be as above. Let $\mathcal{G} = \{V \in L_N(D) : V(x) \geq 0, \|V\|_N = 1\}$. Then among all pairs $(E, V) \in \mathbf{R}^+ \times \mathcal{G}$ for which there exists non-trivial $u \in W_0^{1,n}(D)$ such that*

$$-\Delta_n u = EV|u|^{n-2}u,$$

we have

$$E \geq \frac{1}{K_M|D|},$$

with equality attained by u and V/E, with u, V as in Lemma 3.1. Furthermore, $V(x) = \phi(u(x))$, with ϕ an increasing function whose explicit formula can be found in Lemma 3.1.

As with Corollary 1.3, this result can be compared to the result in [CEP] cited in the introduction.

3.2 A counterexample for $V \in L\log^k L(D),\ k < n-1$.

The purpose of this subsection is to present:

Theorem 3.6 *Let $n \geq 2$ and $0 \leq k < n-1$. Let $N(s) = \int_0^{s/\alpha} \log^k(t+1)\, dt$. For every $\delta > 0$, we can find a non-negative $V_\delta \in L\log^k L(B_1(0))$, and a positive solution $u \in W_0^{1,n}(B_1(0))$ of $-\Delta_n u = V_\delta u^{n-1}$, such that $\|V_\delta\|_N < \delta$.*

Note that when $k = 0$, $V_\delta \in L^1$.

Proof of Theorem 3.6: Our constructed functions will be radial and positive. By (2.9),

$$\Delta_n u(\rho) = (n-1)|u_\rho|^{n-2}(u_{\rho\rho} + \frac{1}{\rho}u_\rho).$$

Let $\delta > 0$ be given and $0 < \epsilon < 1/2$ to be determined later. Let

$$u(\rho) = \begin{cases} a - b\rho^{\frac{n}{n-1}} & \text{if } 0 \leq \rho < \epsilon\ , \\ -\log(\rho) & \text{if } \epsilon \leq \rho \leq 1 \end{cases}$$

where a and b are chosen below so that u is differentiable. Note that $\Delta_n u(\rho) = 0$ for $\epsilon \leq \rho \leq 1$. Continuity at $\rho = \epsilon$ of u_ρ requires $b = \frac{n-1}{n}\epsilon^{-\frac{n}{n-1}}$, and of u requires $a = \frac{n-1}{n} - \log(\epsilon)$. Let C denote a constant which may change from line to line. We define $V_\delta = V$ by the equation $-\Delta_n u = Vu^{n-1}$, which gives $V = 0$ for $\epsilon \leq \rho \leq 1$, and $V \leq \frac{C\epsilon^{-n}}{u^{n-1}(\epsilon)}$ for $0 \leq \rho < \epsilon$. Hence,

$$\int_{B_1(0)} N(|V|)\, dx \leq \frac{C|\log(\epsilon)|^k}{|\log(\epsilon)|^{n-1}}.$$

Let $\lambda = \delta/2$. Observe that $\log(at+1) \leq a\log(t+1)$ for $a > 1$. So, by the integral definition of $N(s/\lambda)$, we have $N(s/\lambda) \leq \lambda^{-(k+1)}N(s) = CN(s)$, for all $s > 0$. Hence

$$\lambda + \frac{\lambda}{K_M|B_1(0)|}\int_{B_1(0)} N(|V|/\lambda)\, dx \leq \delta/2 + C\int_{B_1(0)} N(|V|)\, dx \leq \delta/2 + \frac{C|\log(\epsilon)|^k}{|\log(\epsilon)|^{n-1}}.$$

Choosing ϵ so that $C|\log(\epsilon)|^{k-(n-1)} < \delta/2$, the result follows from (3.3).□

4. Equations with other nonlinear terms

We consider the equation

$$-\Delta_p u = V|u|^\beta u \tag{4.1}$$

where $\beta \geq -1$. This is assumed in the distributional sense, that

$$\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx = \int_D V|u|^\beta u\psi \, dx, \ \forall\psi \in C_0^\infty(D).$$

Theorem 4.1 *Let $1 < p < \infty$ and $r > 1$. Define $\hat{q} = r(\beta+2)/(r-1)$, and assume $\hat{q} \leq p^*$. Let $u \in W_0^{1,p}(D)$ be a nontrivial distributional solution of (4.1) with $V \in L^r(D)$. Then*

$$K^p \|V_+\|_r \ ||u||_{\hat{q}}^{\beta+2-p} \geq 1. \tag{4.2}$$

where $K = K_{\hat{q}p}(D)$. If $\hat{q} < p^*$, equality can be attained in (4.2).

Proof: We prove the result for $p \neq n$; the proof for $p = n$ is almost identical. By Sobolev's inequality, $u \in L^{\overline{q}}(D)$, and hence $u|u|^\beta V \in L^{(p^*)'}(D)$. It follows by Lemma 5.5 and Holder's inequality that

$$||u||_{\hat{q}}^p \leq K^p ||\nabla u||_p^p \leq K^p \int_D |u(x)|^{\beta+2} V_+(x) \, dx \leq K^p ||u||_{\hat{q}}^{\beta+2} ||V_+||_r \tag{4.3}$$

from which (4.2) follows.

If $\hat{q} < p^*$, Lemma 5.3 provides a $u_* \geq 0$, with $||\nabla u_*||_p = 1$, such that $-\Delta_p u_* = cu_*^{\hat{q}-1}$ with

$$c = \|u_*\|_{\hat{q}}^{-\hat{q}} = K^{-p}\|u_*\|_{\hat{q}}^{p-\hat{q}}.$$

So, $-\Delta_p u_* = V u_*^{\beta+1}$, which is (4.1), with $V = V_+ = cu_*^{\hat{q}-2-\beta}$. Thus, $\|V_+\|_r = c\|u_*\|_{\hat{q}}^{\hat{q}-\beta-2}$, which gives equality in (4.2). □

We now consider equations such as $-\Delta_p u = V(x)|\nabla u|^{p-1}$, and give conditions under which $K_{p,p^*}\|V\|_r \geq 1$. More generally, let $u \in W_0^{1,p}(D)$ be a distributional solution of

$$-\Delta_p u = V(x) f(x, u, \nabla u). \tag{4.4}$$

There are numerous works giving *sufficient* conditions for the existence of solutions of equations of this form with D being the unit interval, see [OR], [BGG] and the references therein. In [BGG], the authors prove the existence of multiple solutions for a family of boundary value problems that include (4.4), assuming that f is continuous and non-negative, and V is continuous on $(0,1)$, does not vanish on any open subinterval, and is L^1. The following result partly generalizes Theorem 4.1 and Theorem 3.7 in [DEHL].

Theorem 4.2 *Assume that (4.4) holds, and*

$$|f(x,y,z)| \le |y|^{\beta+1}|z|^{\gamma}, \tag{4.5}$$

with constants $\beta \ge -1$ *and* $\gamma \ge 0$. *Assume* $V \in L^r(D)$ *with* $p \ne n$ *and*

$$\frac{1}{r} + \frac{\beta+2}{p^*} + \frac{\gamma}{p} = 1, \tag{4.6}$$

Then

$$K_{p,p^*}^{p-\gamma}\|V\|_r\|u\|_{p^*}^{2+\beta-p+\gamma} \ge 1. \tag{4.7}$$

This result also holds when $p = n$ *and* p^* *is replaced by* $q < \infty$ *in (4.6) and (4.7).*

Proof: We begin with $p \ne n$. We will assume $\beta > -1$ and $\gamma > 0$. The proof for $\beta = -1$ is similar. The proof for $\gamma = 0$ is similar to the proof of (4.2). By (4.5), (4.6) and $u \in L^{p^*}(D)$, we have $Vf(x,u,\nabla u) \in L^{(p^*)'}$. We can apply Lemma 5.5 and Holder's inequality to get

$$\begin{aligned} \|\nabla u\|_p^p &= \ < Vf, u > \\ &\le \|V\|_r\|f\|_t\|u\|_{p^*}, \end{aligned} \tag{4.8}$$

where t is defined by $\frac{1}{r} + \frac{1}{t} + \frac{1}{p^*} = 1$. Define j by $j(\beta+1)t = p^*$ for $p < n$ and $j = \infty$ for $p > n$. Let $k = p/\gamma t$. Note that $\frac{1}{j} + \frac{1}{k} = t((\beta+1)/p^* + \gamma/p) = 1$, by (4.6). By Holder again and (4.5), we get

$$\begin{aligned} \|f\|_t &\le \||u|^{\beta+1}|\nabla u|^{\gamma}\|_t \\ &\le \||u|^{(\beta+1)t}\|_j^{1/t}\||\nabla u|^{\gamma t}\|_k^{1/t} \\ &= \|u\|_{p^*}^{\beta+1}\|\nabla u\|_p^{\gamma}. \end{aligned} \tag{4.9}$$

Combining Sobolev's inequality, (4.8) and (4.9), we get

$$\|u\|_{p^*}^{p-\gamma} \le K_{p,p^*}^{p-\gamma}\|\nabla u\|_p^{p-\gamma} \le K_{p,p^*}^{p-\gamma}\|V\|_r\|u\|_{p^*}^{2+\beta}.$$

This proves (4.7).

For the case $p = n$, we cannot assume $u \in L^{p^*}$, but we have $u \in L^q$ for all $q < \infty$. Assuming (4.6) holds with some finite q replacing p^*, the proof for this case is the same. □

5. Appendix

5.1 Extremals and their Euler Lagrange equations

Lemma 5.1 *Let* $p > n$ *and* $1 \le q \le \infty$. *Then there is a continuous non-negative Sobolev extremal* $u_* \in W_0^{1,p}(D)$, *with* $\|\nabla u_*\|_p = 1$ *and*

$$\|u_*\|_q = K_{q,p}. \tag{5.1}$$

Proof of Lemma 5.1: We prove the result for $q = \infty$. The proof for $q < \infty$ then follows by using elementary arguments and observing that compact subsets of $L^\infty(D)$ are compact subsets of $L^q(D)$. Let $B_W = \{u \in W_0^{1,p}(D) : \|\nabla u\|_p \leq 1\}$. For $p > 1$, $W^{1,p}(D)$ is reflexive. Since $W_0^{1,p}(D)$ is a closed subspace of $W^{1,p}(D)$, it is also reflexive. Thus B_W is weakly compact with respect to the Sobolev norm. Moreover, the inclusion $W_0^{1,p}(D) \to C^0(D)$ is compact. Let $\{u_n\}$ be a sequence in $W_0^{1,p}(D)$ such that

$$\lim_{n\to\infty} \frac{\|u_n\|_\infty}{\|\nabla u_n\|_p} = K_{\infty,p}.$$

We can assume by scaling that $\|\nabla u_n\|_p = 1$. Since B_W is weakly compact in $W_0^{1,p}(D)$, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ that converges weakly to some $u_* \in B_W$. By the compactness of the inclusion $W_0^{1,p}(D) \to C^0(D)$, there is a subsequence of $\{u_{n_k}\}$, that we label again with $\{u_{n_k}\}$, that converges to some $w \in C^0(D)$ in the strong topology of $C^0(D)$. That is, $\lim_{k\to\infty} \|u_{n_k} - w\|_\infty = 0$. But $u_{n_k} \to u_*$ also in the weak topology of $C^0(D)$, i.e. pointwise and so $u_* = w$ a.e.; consequently, $w \in B_W$ and $\|\nabla w\|_p \leq 1$. We have

$$K_{\infty,p} = \lim_{k\to\infty} \|u_{n_k}\|_\infty = \|w\|_\infty,$$

and so $\dfrac{\|w\|_\infty}{\|\nabla w\|_p} \geq K_{\infty,p}$. But recall that $w = u_* \in W_0^{1,p}(D)$, and so $\dfrac{\|w\|_\infty}{\|\nabla w\|_p} \leq K_{\infty,p}$ proving (5.1) for $u_* = w$. If u_* is not already non-negative, we can replace it by $|u_*|$, with no effect on (5.1) (see [LL]). $\square$

Recall that $p^* = \frac{np}{n-p}$ for $1 < p < n$ and $p^* = \infty$ for $n \leq p$.

Lemma 5.2 *Suppose* $1 < p \leq n$ *and* $1 < q < \bar{q}$. *Then there is a non-negative Sobolev extremal* $u_* \in W_0^{1,p}(D)$ *with* $\|\nabla u_*\|_p = 1$ *and*

$$\|u_*\|_q = K_{q,p}.$$

The proof of this result is a straightforward adaptation of the proof of Lemma 5.1 (also see [DEHL]), and is left to the reader.

Lemma 5.3 *Let* $1 < p < \infty$ *and* $p \leq q < p^*$, *and let* u_* *be a Sobolev extremal as in Lemma 5.1 or Lemma 5.2. Then*

$$-\Delta_p u_* = \frac{u_*^{q-1}}{\|u_*\|_q^q}. \tag{5.2}$$

The proof of Lemma 5.3 is similar to ([DEHL], Lemma A.2), and is left to the reader.

5.2 Green's identities for divergence and Orlicz forms.

Lemma 5.4 *Let* $u \in W_0^{1,p}(D)$, *with* $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = V|u|^{p-2}u$ *in the distribution sense. Assume either*

A) $p < n$ *and* $V \in L^{n/p}(D)$, *or*

B) $p \geq n$ and $V \in L^r(D)$ for some $r > 1$, or
C) $n = p$, u is real-valued and $V \in L\log^{n-1} L(D)$. Then

$$\int_D |\nabla u|^p dx = \int_D V|u|^p dx. \tag{5.3}$$

Proof. By assumption

$$\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\psi dx = \int_D V|u|^{p-2}u\psi \; dx \tag{5.4}$$

for every $\psi \in C_0^\infty(D)$. Let $\{\psi_j\}$ be a sequence of functions in $C_0^\infty(D)$ that converges to $\overline{u}$ in $W_0^{1,p}(D)$. Then

$$\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\psi_j \, dx - \int_D |\nabla u|^p \, dx \leq ||\nabla\psi_j - \nabla\overline{u}||_p|||\nabla u|^{p-1}||_{p'} \to 0.$$

To complete the proof of (5.3), it suffices to show that $V|u|^{p-2}u\psi_j$ converges to $V|u|^p$ in $L^1(D)$ in each case. Assume A), that $p < n$ and $V \in L^{n/p}(D)$. By Sobolev's inequality, ψ_j converges to $\overline{u}$ in $L^{\frac{pn}{n-p}}(D) = L^{p^*}(D)$. By Holder's inequality,

$$\int_D |V|u|^{p-2}u\psi_j - V|u|^p| \, dx \leq ||V||_{n/p}||u^{p-1}||_{p^*/(p-1)}\|\psi_j - \overline{u}\|_{p^*} \to 0.$$

The proof in case B) is similar. So, assume C), and without loss of generality, that $\|\nabla u\|_n = 1$. Let $\{\psi_m\} \subset C_0^\infty(D)$ converge to u in $W_0^{1,n}(D)$. We can choose $\lambda_m \downarrow 0$ so that $\|\nabla(u - \psi_m)\|_n\lambda_m^{-1} \to 0$. The Moser-Trudinger inequality implies, for n large,

$$\int_D e^{\left(\alpha_n \frac{|u-\psi_m|}{\lambda_m}\right)^{n/(n-1)}} dx \leq \int_D e^{\left(\alpha_n \frac{|u-\psi_m|}{\|\nabla(u-\psi_m)\|_n}\right)^{n/(n-1)}} dx < (C_n + 1)|D| < \infty,$$

with C_n independent of u and ψ_m. A similar inequality holds when $\frac{u-\psi_m}{\lambda_m}$ is replaced by u.

Let M and N be the functions defined in (3.1). We have $M(t) \leq Ce^{\alpha_n^{n/(n-1)}t^{1/(n-1)}}$ and $N(t) \sim t\log^{n-1}(t)$ for large t. It follows from $V \in L\log^{n-1} L(D)$ that

$$\int_D N(|V|) < \infty. \tag{5.5}$$

Using Young's inequality, the inequality $|ab| \leq \frac{n-1}{n}a^{n/(n-1)} + \frac{1}{n}b^n$, and Hölder's inequality:

$$\begin{aligned}
\int_D \left|u^{n-1} V(u-\psi_m)\right| dx &= \lambda_m \int_D |\frac{u^{n-1}(u-\psi_m)}{\lambda_m}V|dx \\
&\leq \lambda_m \int_D M\left(\frac{|u|^{n-1}(u-\psi_m)|}{\lambda_m}\right) dx + \lambda_m \int_D N(|V|)dx \\
&\leq C\lambda_m \int_D \exp\left(\alpha_n^{\frac{n}{n-1}}|u|\left(\frac{|u-\psi_m|}{\lambda_m}\right)^{1/(n-1)}\right) dx + \lambda_m \int_D N(|V|)dx \\
&\leq C\lambda_m \int_D e^{\frac{n-1}{n}|\alpha_n u|^{\frac{n}{n-1}}} e^{\frac{1}{n}(\alpha_n|u-\psi_m|\lambda_m^{-1})^{\frac{n}{n-1}}} dx + \lambda_m \int_D N(|V|)dx \\
&\leq \lambda_m \left\{C\left(\int_D e^{|\alpha_n u|^{\frac{n}{n-1}}} dx\right)^{\frac{n-1}{n}} \left(\int_D e^{(\alpha_n \frac{|u-\psi_m|}{\lambda_m})^{\frac{n}{n-1}}} dx\right)^{\frac{1}{n}} + \int_D N(|V|)dx\right\}
\end{aligned}$$

Thus $u\psi_m|u|^{n-2}V$ converges in $L^1(D)$ to $|u|^nV$.□

The next lemma is used in Section 4. It contains Lemma 5.4 parts A) and B) as special cases.

Lemma 5.5 *Let $u \in W_0^{1,p}(D)$ be a solution in the distribution sense of $-\Delta_p u = F$. Assume $F \in L^{(p^*)'}(D)$ when $n \neq p$, and $F \in L^r(D)$ for some $r > 1$ when $n = p$. Then,*

$$\int_D |\nabla u|^p dx = \int_D F\overline{u} dx. \tag{5.6}$$

Proof. We have $\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\psi dx = \int_D F\psi \, dx$ for every $\psi \in C_0^\infty(D)$. The rest is similar to the proof of Lemma 5.4, part A).

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