

# Minimal support results for Schrödinger equations

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## Abstract

We consider a number of linear and non-linear boundary value problems involving generalized Schrödinger equations. The model case is  $-\Delta u = Vu$  for  $u \in W_0^{1,2}(D)$  with  $D$  a bounded domain in  $\mathbf{R}^n$ . We use the Sobolev embedding theorem, and in some cases the Moser-Trudinger inequality and the Hardy-Sobolev inequality, to derive necessary conditions for the existence of nontrivial solutions.

These conditions usually involve a lower bound for a product of powers of the norm of  $V$ , the measure of  $D$ , and a sharp Sobolev constant. In most cases, these inequalities are best possible. We also derive an explicit formula for the Sobolev constant on an interval.

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## 1. Introduction

We show that solutions of certain second order elliptic differential equations cannot vanish on the boundary of arbitrarily small domains. Our first example is the Schrödinger equation

$$-\Delta u = V(x)u, \quad x \in D, \quad (1.1)$$

where  $\Delta = \sum_{j=1}^n \partial_{x_j}^2$  is the Laplace operator, and  $D$  is, here and throughout the paper, a bounded domain of  $\mathbf{R}^n$ . Unless stated otherwise,  $u \in W_0^{1,2}(D)$  will be a solution in the distribution sense of (1.1), i.e.,

$$\int_D \nabla u(x) \cdot \nabla \psi(x) dx = \int_D V(x)u(x)\psi(x) dx$$

for every  $\psi \in C_0^\infty(D)$ .

We say that a solution  $u$  of (1.1) is *trivial* if  $u(x) = 0$  almost everywhere (a.e.) on  $D$ . Unless otherwise stated, we assume  $u$  is nontrivial and complex valued, and that (1.1) holds in the sense of distributions.

$V$  is often referred to as the *potential* of the equation. Although several of our proofs work for complex valued  $V$ , we will assume throughout this paper that  $V$  is real-valued, and we let  $V_+ = \max\{V, 0\}$ . We denote by  $\|V\|_r = \|V\|_{L^r(D)}$  the usual Lebesgue norm, and say  $u$  is in the Sobolev space  $W_0^{1,2}(D)$  if  $u$  is in the closure of  $C_0^\infty(D)$  with respect to the norm  $\|\nabla u\|_2$ .

Let  $K_q(D)$  be the operator norm of the Sobolev embedding  $W_0^{1,2}(D) \rightarrow L^{2q}(D)$ . That is,

$$K_q(D) = \sup_{u \neq 0} \frac{\|u\|_{2q}}{\|\nabla u\|_2}. \quad (1.2)$$

For  $n > 2$ , let  $2\bar{q} = \frac{2n}{n-2}$  be the critical index in the Sobolev embedding theorem. That is,  $K_q(D)$  is finite for  $q \leq \bar{q}$  and infinite otherwise. When  $n \leq 2$ , let  $\bar{q} = \infty$ . Note that when  $n = 2$  and  $q = \bar{q} = \infty$ , then  $K_q(D) = \infty$ , an exceptional case addressed carefully in Section 2. When  $n = 2$ , we will assume  $q < \infty$  unless specified otherwise.

We often assume that  $V \in L^r$ , where  $r = q^*$  is the Hölder conjugate exponent to  $q$ . So  $\frac{1}{q} + \frac{1}{r} = 1$ , with the convention that  $1^* = \infty$ . We will abuse notation slightly, and write  $r \geq \frac{n}{2}$  to mean  $\max\{1, \frac{n}{2}\} \leq r \leq +\infty$ . Note that when  $r \geq \frac{n}{2}$ , then  $r^* \leq \frac{n}{n-2} = \bar{q}$ .

Our first result is our simplest, and is central to the rest of the paper.

**Theorem 1.1** *Suppose that the Schrödinger equation (1.1) has a nontrivial solution  $u \in W_0^{1,2}(D)$  for some  $V \in L^r(D)$ , with  $r > \frac{n}{2}$  and  $q = r^*$ . Then,*

$$K_q^2(D) \|V_+\|_r \geq 1. \quad (1.3)$$

Denote by  $u_*$  an extremal<sup>1</sup> in the inequality (1.2). With  $u = u_*$ , and

$$V(x) = \frac{\|\nabla u_*\|_2^2}{\|u_*\|_{2q}^{2q}} |u_*(x)|^{2q-2}, \quad (1.4)$$

equality is attained in (1.1) and in (1.3).

When  $q < \bar{q}$ ,  $K_q(D)$  depends on the volume as well as the shape of  $D$ . Let  $K_q^*$  be the Sobolev constant associated with the ball of volume 1. It is well-known that  $K_q^* = \max_{|D|=1} K_q(D)$ . A simple dilation argument proves the following

**Theorem 1.2** *Under the assumptions of Theorem 1.1,*

$$(K_q^*)^2 |D|^{\frac{2}{n} - \frac{1}{r}} \|V_+\|_r \geq 1. \quad (1.5)$$

So, if the Schrödinger equation (1.1) has nontrivial solutions in  $W_0^{1,2}(D)$ , and  $\|V\|_r$  is fixed, then  $|D|$  cannot be too small; we say that the solutions have a *minimal support property*.

The proofs of Theorem 1.1, and many others in this paper, follow a pattern from Theorem 4.1 in [DH1], which we refer to as the *minimal support sequence*. For example, to prove (1.3) for  $u \in C_0^2(D)$ , we use Sobolev's inequality (1.2), Green's identity, and Hölder's inequality:

$$\begin{aligned} \|u\|_{2q}^2 &\leq K_q^2(D) \int_D |\nabla u|^2 dx = -K_q^2(D) \int_D \bar{u} \Delta u dx \\ &= K_q^2(D) \int_D |u|^2 V dx \leq K_q^2(D) \int_D |u|^2 V_+ dx \\ &\leq K_q^2(D) \|u\|_q \|V_+\|_r = K_q^2(D) \|u\|_{2q}^2 \|V_+\|_r \end{aligned} \quad (1.6)$$

which implies (1.3). If  $u = u_*$  is an extremal for the Sobolev inequality (1.2), the first inequality in (1.6) is an equality. We prove in Lemma 5.2 that  $u_*$  solves (1.1), with  $V = V_+$

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<sup>1</sup> $u_*$  is an extremal for (1.2) if  $K_q(D) = \frac{\|u_*\|_{2q}}{\|\nabla u_*\|_2}$ .

as in (1.4). Remarkably, this result also makes Hölder's inequality into an equality. So, (1.3) is an equality too. This apparent coincidence has an explanation; the solutions of  $-\Delta u = Vu$  minimize certain energy functionals. Since  $u_*$  solves two similar optimization problems, it has certain unexpected properties; see [H]. We complete the proof of Theorem 1.1 in Section 2.1.

Theorem 1.1 can be interpreted as a necessary condition for zero to be an eigenvalue for the operator  $-\Delta - V$  with quadratic form domain  $W_0^{1,2}(D)$ .

**Corollary 1.3** *Suppose that the differential equation*

$$-\Delta u - Vu = Eu$$

*has a nontrivial solution in  $W_0^{1,2}(D)$  for some constant  $E \leq 0$ . Suppose  $V$ ,  $r$ ,  $q$  and  $K_q(D)$  are as in Theorem 1.1. Then*

$$K_q^2(D) \|V_+\|_r \geq 1. \tag{1.7}$$

*If  $n \geq 3$  and  $r = \frac{n}{2}$ , then  $K_q^2 \|V_+\|_{\frac{n}{2}} > 1$ .*

This result for  $r > \frac{n}{2}$  follows immediately from Theorem 1.1 by noting that  $(V + E)_+(x) \leq V_+(x)$ ; the case  $n \geq 3$  and  $r = \frac{n}{2}$  follows similarly from Theorem 2.1.

The analogue of Corollary 1.3 for  $D = \mathbf{R}^n$  has a long history in the mathematical physics community, motivated by questions of the existence of bound states, i.e.,  $L^2$  eigenvalues, for the Schrödinger operator in  $\mathbf{R}^n$ . Specifically, consider

$$-\Delta u - Vu = Eu, \quad u \in W^{1,2}(\mathbf{R}^n), \tag{1.8}$$

with eigenvalue  $E < 0$ . For fixed  $V$ , let  $\tilde{N}$  be the number of negative eigenvalues of  $-\Delta - V$ . The following inequality is due to Cwikel [Cw], Lieb [L], and Rozenblum [Roz]:

$$C_n \|V_+\|_{\frac{n}{2}} \geq \tilde{N}, \quad n \geq 3, \tag{1.9}$$

where  $C_n$  depends only on  $n$ . For more on the values of  $C_n$ , and also for the cases  $n = 1$  and  $n = 2$ , the reader is referred to the survey on bound states by Hundertmark [Hun].

A more abstract version of the Cwikel-Lieb-Rozenblum inequality, which applies on a bounded domain  $D$ , as does our Corollary 1.3, is derived in Theorem 2.1 in [FLS]. It implies

$$C \|V_+\|_r \geq \tilde{N}, \quad r \geq \frac{n}{2}, \tag{1.10}$$

where the best constant  $C$  is unknown, but satisfies  $K_q^2(D) \leq C \leq e^{1-\frac{1}{r}} K_q^2(D)$ . If there is exactly one negative eigenvalue, then  $\tilde{N} = 1$ , and thus  $C \|V_+\|_r \geq 1$ ; in this special case, the bound (1.7) improves on (1.10). It is not clear whether (1.10) can be compared with (1.3), since Theorem 1.1 involves a zero eigenvalue.

We now consider the lower bound  $r > \frac{n}{2}$  in Theorems 1.1 and 1.2. When  $n \geq 3$  and  $r = \frac{n}{2}$ , (1.2) still holds, but equality is not attained on any proper subset  $D \subset \mathbf{R}^n$ . So, (1.3)

in Theorem 1.1 (and likewise (1.6) and (1.5)) still holds with the same proof, but equality cannot be attained. However, the estimate is still sharp; see Theorem 2.1 in Section 2.2. These results do not extend to  $r < \frac{n}{2}$ ; see Theorem 2.2.

When  $n = 1$ , the critical case is  $r = 1$ . Theorem 1.1 is still valid, but to attain equality in (1.3), we must allow  $V$  to be a finite measure rather than a  $L^1$  function; see Section 4.1.

When  $n = 2$ , Theorem 2.2 shows that Theorem 1.1 does not extend to  $r = \frac{n}{2} = 1$ . The minimal support sequence fails because  $W_0^{1,2}(D)$  does not embed into  $L^\infty(D)$ . However, we prove an analogue of Theorem 1.1 when  $V$  is in the Orlicz space  $L \log L(D)$ . Our main result in Section 2.4 uses a norm  $\|\cdot\|_{N_D}$  for  $L \log L(D)$  defined by (2.11).

**Theorem 1.4** *Assume that (1.1) has a nontrivial solution with  $V$  in  $L \log L(D)$  with  $D \subset \mathbf{R}^2$ . Then*

$$\frac{C_2|D|}{4\pi} \|V_+\|_{N_D} \geq 1, \quad (1.11)$$

where  $C_2$  is the constant of the Moser-Trudinger inequality (2.9).

Let  $u_*$  be an extremal for (2.9), normalized by  $\|\nabla u_*\|_2 = 1$ . Then for

$$V = \frac{e^{4\pi|u_*(x)|^2}}{\int_D |u_*(x)|^2 e^{4\pi|u_*(x)|^2} dx}, \quad (1.12)$$

equality holds in (1.11).

We have the following analogue of Corollary 1.3 in this case.

**Corollary 1.5** *Let  $u \in W_0^{1,2}(D)$  be a nontrivial solution of*

$$-\Delta u - Vu = Eu \quad (1.13)$$

with  $E \leq 0$  and  $V \in L \log(L)$ . Then (1.11) holds.

To the best of our knowledge, this result is new. For other results relating the spectrum to  $V$  on bounded domains, see [Hen]. In Section 4 we consider results for an interval  $D = (-b, b) \subset \mathbf{R}^1$ . As an application of our methods, we find an explicit formula for  $K_q(D)$  for  $q \in (1, \infty)$ ; to the best of our knowledge, it is original. For related results, see [TMH], [AS].

**Theorem 1.6** *For  $1 < q < \infty$ , define  $K_q = K_q(-b, b)$  as in (1.2). Then  $K_q = b^{\frac{1}{2q} + \frac{1}{2}} M_q$ , where*

$$M_q = \frac{2\sqrt{q} \Gamma\left(\frac{q+1}{2q}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{2q}\right)} \left(\frac{q+1}{2}\right)^{\frac{q-1}{2q}}. \quad (1.14)$$

Theorem 1.1 extends, in part, the main result in [DH2]; two of the authors proved that if  $V \in L^\infty(D)$ , and if  $u \in C_0(\overline{D})$  is a nontrivial solution of (1.1), then

$$|D|^{\frac{2}{n}} \left( j^{-1} \omega_n^{-\frac{1}{n}} \right)^2 \cdot \|V\|_\infty \geq 1,^2 \quad (1.15)$$

where  $j$  is the first positive zero of the Bessel function  $J_{\frac{n}{2}-1}$ . Equality is attained when  $u = u_*(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|)$ . The proof in [DH2], which is quite different from the arguments appearing here, compares the level sets of  $u(x)$  and  $u_*(x)$ .

\*\* new 3/22 \*\* By comparing (1.15) and (1.5), we can see at once that the Sobolev constant  $K_1^*$  associated with the ball of volume 1 is  $K_1^* = (j\omega_n^{\frac{1}{n}})^{-1}$ . Since  $K_1^*$  is the reciprocal of the first eigenvalue of  $-\Delta$  on the ball of volume 1, its value is well known, but it is interesting to observe how this explicit expression follows from our theorems. <sup>3</sup> \*\*

Our Theorem 1.1, as well as many other theorems in this paper, can be viewed as a unique continuation result for solutions of the equation under consideration. That is, if  $u$  is a solution of (1.1) that vanishes on the boundary of  $D$ , and if  $|D|$  is too small, then  $u \equiv 0$  in  $D$ . In unique continuation problems the zero set of  $u$  is usually assumed to be an open set, or a point, but in our case, it may be an  $(n-1)$ -dimensional boundary. Our assumption that  $V \in L^r(D)$ , with  $r > \frac{n}{2}$ , is also critical in these problems.

This paper is organized as follows: in Section 2 we prove necessary conditions for the existence of nontrivial solutions for the Schrödinger equation (1.1) with various assumptions on  $V$ . In Section 3, we consider similar questions for other well-known linear and nonlinear second order equations. Minimal support problems in  $\mathbf{R}^1$  are handled separately in Section 4. We have collected some technical lemmas, perhaps not entirely new, into an appendix.

## 2. The Schrödinger equation.

In this section we prove necessary conditions for the existence of nontrivial solutions in  $W_0^{1,2}(D)$  of the Schrödinger equation  $-\Delta u = V(x)u$ . We also consider potentials which do not necessarily belong to  $L^r(D)$  with  $r > \frac{n}{2}$ , but are dominated by a Hardy potential, or belong to  $L^{\frac{n}{2}}(D)$  or to an Orlicz space.

### 2.1 Completion of the proof of Theorem 1.1

We used the minimal support sequence in the introduction to prove that  $K_q^2(D) \|V_+\|_r \geq 1$  when  $u \in C_0^2(D)$ . When  $u \in W_0^{1,2}(D)$ , we cannot apply the standard Green's identity, but

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<sup>2</sup>L. was "  $|D| \cdot \|V\|_\infty^{\frac{n}{2}} \geq j^n \omega_n$ , which I have rewritten to match the inequality in CorY

<sup>3</sup>L. removed " This formula is valid for every  $n \geq 1$ . When  $n = 1$ , the first positive zero of  $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$  is  $\frac{\pi}{2}$ , and  $\omega_1 = 2$ ; we obtain the well known formula  $K_1^* = \pi^{-1}$ , a special case of Theorem 1.6."

we use instead the identity (5.6) in Lemma 5.4:

$$\int_D |\nabla u|^2 dx = \int_D |u|^2 V dx.$$

All other inequalities in the minimal support sequence hold also when  $u \in W_0^{1,2}(D)$ , and so (1.6) is proved.

We now prove that equality can occur in (1.3). Since  $q = r^* < \bar{q}$ , there exists an extremal  $u_* \geq 0$  for the Sobolev inequality (1.2); this result is probably known, but we prove it in the appendix as Lemma 5.1 for completeness. Furthermore, by Lemma 5.2,  $u_*$  is a solution in the distribution sense of (1.1) with  $V(x) = \frac{\|\nabla u_*\|_2^2 |u_*(x)|^{2q-2}}{\|u_*\|_{2q}^{2q}}$ . Since  $(q-1)r = q$ ,

$$\left( \int_D |u_*(x)|^{(2q-2)r} dx \right)^{\frac{1}{r}} = \|u_*\|_{2q}^{2q-2}$$

and

$$K_q^2(D) \|V\|_r = K_q^2(D) \frac{\|\nabla u_*\|_2^2}{\|u_*\|_{2q}^2} = 1. \quad \square$$

## 2.2 A critical case for $n \geq 3$ : $V \in L^{\frac{n}{2}}$

In this section, we assume  $n \geq 3$  and  $V \in L^r(D)$ , where  $r = \frac{n}{2}$ . This assumption on  $V$  is weaker than the assumption  $r > \frac{n}{2}$  in Theorem 1.1, and  $r = \frac{n}{2}$  may be regarded as a critical case; see also the next two subsections. In Proposition 2.2, we show that no minimal support result is possible with smaller  $r$  by providing explicit counterexamples. We also briefly discuss the case  $n = 2$  there, with more about that in Sections 2.3 and 2.4. For  $r = \frac{n}{2}$ , we have  $r^* = \bar{q} = \frac{n}{n-2}$ . The Sobolev inequality

$$\|u\|_{2\bar{q}} < K_{\bar{q}} \|\nabla u\|_2. \quad (2.1)$$

is strict (since  $D \neq \mathbb{R}^n$ ) and dilation invariant. So,  $K_{\bar{q}}$  and the corresponding minimal support sequence are independent of  $|D|$ . In a celebrated theorem, Talenti (see [T]) proved that  $K_{\bar{q}} = (n(n-2)\pi)^{-\frac{1}{2}} \left( \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right)^{\frac{1}{n}}$ . In the next theorem, instead of a minimal support result, we prove a "minimal potential result".

**Theorem 2.1** *Suppose  $V \in L^{\frac{n}{2}}(D)$ , with  $n \geq 3$ . If (1.1) has nontrivial solutions in  $W_0^{1,2}(D)$ , and  $\bar{q}$  and  $K_{\bar{q}}$  are as above,*

$$K_{\bar{q}}^2 \|V_+\|_{\frac{n}{2}} > 1 \quad (2.2)$$

*and the constant 1 is sharp.*

*Proof.* The minimal support sequence (1.6) proves (2.2). In this case, we get a strict inequality because (2.1) is strict. Now we prove that (2.2) does not hold if 1 is replaced by any larger constant. Since (2.1) is invariant by dilation, the constant  $K_{\bar{q}}$  is independent of  $D$  (and is also the same for  $\mathbf{R}^n$ ). In our proof we will define a suitable large disk  $D$  and  $u \in W_0^{1,2}(D)$  such that  $K_{\bar{q}}^2 \|V\|_{\frac{n}{2}} \approx 1$ . Let  $v(\rho) = (1 + \rho^2)^{\frac{2-n}{2}}$ , with  $\rho = |x|$ . Talenti showed in [T] that this  $v$  gives equality in (2.1) on  $\mathbf{R}^n$ . Define  $V_v(\rho) = -\Delta v(\rho)/v(\rho)$ .

Recalling that the Laplacian of a radial function  $u$  in  $\mathbf{R}^n$  is  $\Delta u = u_{\rho\rho} + \frac{(n-1)}{\rho}u_\rho$ , it is easy to verify that  $V_v(\rho) = \frac{(n-2)n}{(\rho^2+1)^2}$  and that

$$K_{\bar{q}}^n \int_{\mathbf{R}^n} |V_v|^{\frac{n}{2}} dx = 1. \quad (2.3)$$

Indeed,

$$\begin{aligned} \|V_v\|_{\frac{n}{2}}^{\frac{n}{2}} &= (n(n-2))^{\frac{n}{2}} |S^{n-1}| \int_0^\infty \rho^{n-1} (1 + \rho^2)^{-n} d\rho \\ &= \frac{1}{2} (n(n-2))^{\frac{n}{2}} |S^{n-1}| \int_0^1 (t - t^2)^{\frac{n}{2}-1} dt = (n(n-2))^{\frac{n}{2}} \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} = K_{\bar{q}}^{-n}. \end{aligned}$$

We have used the substitution  $(1 + \rho^2)^{-1} = t$ . Let  $D' = B_R(0)$ , where  $R > 0$  will be specified later. We will define a compactly supported, non-negative function  $u$  by perturbing  $v$  on  $D \setminus D'$ , while keeping  $K_{\bar{q}}^2 \|V_u\|_{\frac{n}{2}} \approx 1$ . Let

$$u(\rho) = \begin{cases} v(\rho) & \text{if } 0 \leq \rho \leq R, \\ a\rho + b & \text{if } R < \rho < R+1, \\ c\rho^{2-n} + d & \text{if } R+1 \leq \rho \leq \hat{R} \end{cases}$$

where  $a, b, c, d$  and  $\hat{R}$  are chosen below so that  $u$  is differentiable, and vanishes at  $\rho = \hat{R} = \partial D$ . Note that  $u$  is harmonic for  $\rho > R+1$ . Choose  $a = v'(R) = (2-n)R(R^2+1)^{-\frac{n}{2}}$ , and  $b = v(R) - aR = (R^2+1)^{-\frac{n}{2}}(R^2(n-1)+1)$ , which makes  $u$  differentiable at  $|x| = R$ .

In what follows,  $C$  will denote a positive constant that may change from line to line, but is always independent of  $R$ . When  $R < \rho < R+1$ ,  $u(\rho) = a\rho + b \geq CR^{2-n}$ , and  $|\Delta u(\rho)| = |a|(n-1)\rho^{-1} \leq C(n-1)(n-2)(1+R^2)^{-\frac{n}{2}} \leq CR^{-n}$ . Thus,

$$\int_{R \leq \rho < R+1} |V_u(x)|^{\frac{n}{2}} dx \leq C \int_R^{R+1} |R^{-2}|^{\frac{n}{2}} \rho^{n-1} d\rho \leq CR^{-1}. \quad (2.4)$$

Next, choose  $c = R(R+1)^{n-1}(R^2+1)^{-\frac{n}{2}}$ , and  $d = ((1-n)R+1)(R^2+1)^{-\frac{n}{2}}$  so that  $u$  is differentiable at  $|x| = \rho = R+1$ . Since  $d < 0$ , there exists  $\hat{R} > R+1$  for which  $u(\hat{R}) = 0$ . We let  $D = B_{\hat{R}}(0)$ . Then  $u \in W_0^{1,2}(D)$ . Since  $u$  is harmonic for  $\rho > R+1$ , (2.3) and (2.4) imply

$$K_{\bar{q}}^n \int_D |V_u|^{\frac{n}{2}} dx \leq 1 + \frac{C}{R} \rightarrow 1$$

as  $R \rightarrow \infty$ .  $\square$

The following constructions show that the conclusions of Theorems 1.1 and 1.2 do not hold when  $r < \frac{n}{2}$ , nor when  $r = 1$  and  $n = 2$ .

**Proposition 2.2** *Let  $n \geq 3$  and  $r < \frac{n}{2}$  (or  $n = 2$  and  $r = 1$ ). For every  $\epsilon > 0$ , we can find a non-negative  $V_\epsilon \in L^r(B_1(0))$ , and a nontrivial solution  $u \in W_0^{1,2}(B_1(0))$  of  $-\Delta u = V_\epsilon u$ , such that  $\lim_{\epsilon \rightarrow 0} \|V_\epsilon\|_r = 0$ .*

*Proof.* Suppose  $n = 3$ . Let  $\epsilon > 0$  be small. For  $\epsilon \leq \rho \leq 1$ , set  $u(\rho) = \rho^{-1} - 1$ , so  $u(1) = 0$  and  $u$  is harmonic. For  $0 \leq \rho \leq \epsilon$ , set  $u(\rho) = a - b\rho^2$ . We choose  $b = (2\epsilon^3)^{-1}$  so that  $u'(\rho)$  is continuous at  $\epsilon$ , and we chose  $a = \frac{3}{2\epsilon} - 1$ , so that  $u(\rho)$  is continuous at  $\epsilon$ . So,  $V_\epsilon$  is supported on  $B_\epsilon(0)$ , and there  $\Delta u = -6b$  and  $u \geq C\epsilon^{-1}$  (since  $b\rho^2 \leq b\epsilon^2 = \frac{1}{2\epsilon}$ ). Hence,

$$|V_\epsilon| \leq C\epsilon^{-2},$$

so for  $r < \frac{3}{2}$ ,

$$\|V_\epsilon\|_r \leq C\epsilon^{3-2r} \rightarrow 0, \text{ as } \epsilon \rightarrow 0.$$

For larger  $n$ , we set  $u(\rho) = \rho^{2-n} - 1$  for  $\rho > \epsilon$  instead, with a similar proof.

For  $n = 2$ , we set  $u(\rho) = -\ln(\rho)$  for  $\epsilon \leq \rho \leq 1$  which is harmonic. For  $\rho < \epsilon$ , set  $u(\rho) = a - b\rho^2$ . We choose  $b = (2\epsilon^2)^{-1}$  so that  $u'(\rho)$  is continuous at  $\epsilon$ , and we chose  $a = \frac{1}{2} - \ln(\epsilon)$ , so that  $u(\rho)$  is continuous at  $\epsilon$ . Near 0,  $\Delta u = -4b$  and  $u > -\ln(\epsilon)$ ; thus,  $0 \leq V_\epsilon(r) < \frac{2}{\epsilon^2 \ln(\epsilon^{-1})}$ , and  $\|V_\epsilon\|_1 \leq \frac{C}{\ln(\epsilon^{-1})} \rightarrow 0$ .  $\square$

*Remark:* When  $n = 2$ , the proof of Proposition 2.2 shows that  $|x|^2 V_\epsilon(x) \leq \frac{2}{\ln(\epsilon^{-1})} \rightarrow 0$ , as  $\epsilon \rightarrow 0$ . See also the remark following Theorem 2.3.

## 2.3 Hardy potentials

We now prove minimal potential results for solutions of the Schrödinger equation with pointwise bounds on  $|V|$ , but no longer assuming  $V \in L^{\frac{n}{2}}(D)$ . For example, we study  $V = C|x|^{-2}$ , which is known as a *Hardy potential*. Let  $\text{dist}(x) = \inf\{|x - y|, y \in \partial D\}$ .

**Theorem 2.3** *Suppose  $n \geq 2$ , and that a measurable  $V$  satisfies one of these on  $D$ :*

i)

$$|V(x)| \leq \left(\frac{n-2}{2}\right)^2 |x|^{-2}; \quad \text{or} \quad (2.5)$$

ii)  $D$  is convex with piecewise-smooth boundary, and

$$|V(x)| \leq \frac{1}{4} \text{dist}(x)^{-2}. \quad (2.6)$$

Then (1.1) has no nontrivial solutions in  $W_0^{1,2}(D)$ .



We do not assume that  $0 \in D$ . Also, note that when  $n = 2$ , (2.5) reiterates that (1.1) has only trivial solutions when  $V \equiv 0$ .

When  $n = 2$  and (2.5) is replaced by  $|V(x)| \leq C|x|^{-2}$ , for some positive constant  $C$ , the remark after the proof of Proposition 2.2 shows that (1.1) can have nontrivial solutions.

*Proof.* Suppose  $V$  satisfies (2.5) and that (1.1) has a nontrivial solution  $u \in W_0^{1,2}(D)$ . We use a variation of the classical Hardy Sobolev inequality (see [BV]):

$$\int_D |\nabla u(x)|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_D \frac{|u(x)|^2}{|x|^2} dx \geq C(D) \|u\|_2^2 > 0. \quad (2.7)$$

By Green's identity (Lemma 5.4, with  $a \equiv b \equiv 1$ ) and the above,

$$\begin{aligned} \left(\frac{n-2}{2}\right)^2 \int_D \frac{|u(x)|^2}{|x|^2} dx &< \int_D |\nabla u(x)|^2 dx \\ &= \int_D V(x) |u(x)|^2 dx \leq \int_D |V(x)| |u(x)|^2 dx \end{aligned}$$

which contradicts (2.5).

Assume now that  $V$  satisfies (2.6) and that  $u$  is a nontrivial solution of (1.1). Using an inequality in [BFT],

$$\int_D |\nabla u(x)|^2 dx - \frac{1}{4} \int_D \frac{u^2(x)}{\text{dist}(x)^2} dx \geq \frac{1}{4 \text{diam}^2(D)} \int_D |u(x)|^2 dx > 0. \quad (2.8)$$

By Lemma 5.4 and the above,

$$\begin{aligned} \frac{1}{4} \int_D \frac{|u(x)|^2}{\text{dist}(x)^2} dx &< \int_D |\nabla u(x)|^2 dx \\ &= \int_D V |u(x)|^2 dx \leq \int_D |V(x)| |u(x)|^2 dx \end{aligned}$$

contradicting (2.6).  $\square$

*Remark.* In the recent paper [FL], the authors improve the inequality (2.8), and prove a Cwikel-Lieb-Rozenblum type inequality for the negative eigenvalues of  $H = -\Delta - (2D_\Omega)^{-2} + V$ , where  $D_\Omega$  is a function that can be replaced by  $\text{dist}(x)$  when  $D$  is convex. They also observe, using a minimal support sequence similar to ours, that  $H$  will have no negative eigenvalues if  $\|V_-\|_{\frac{n}{2}}$  is sufficiently small. For other results related to the first and second parts of our theorem, see [Da],[KO].

## 2.4 A critical case for $n=2$ : $V \in L \log L$ .

In this section, we prove Theorem 1.4 by proving the equivalent Theorem 2.4 below. We have observed that Theorem 1.1 does not hold when  $n = 2$  and  $V \in L^1(D)$ . Here, we

assume  $V$  in the Orlicz space  $L \log L(D)$ , defined as the set of measurable functions  $f$  such that  $\int_D |f|(1 + \log^+ |f|) dx$  is finite. We will use the Moser-Trudinger inequality (see [M]) as a substitute for (1.2); it is

$$\int_D \left( e^{4\pi \left( \frac{|u(x)|}{\|\nabla u\|_2} \right)^2} - 1 \right) dx \leq C_2 |D|, \quad u \in W_0^{1,2}(D), \quad (2.9)$$

where the constant  $C_2$  does not depend on  $u$  or  $D$ . Let  $M(x) = e^x - 1$ , and

$$N(y) = \begin{cases} y \log(y) - y + 1, & \text{if } y \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.10)$$

Following [KR], we set for  $V \in L \log L(D)$

$$\|V\|_{N_D} = \inf \left\{ \lambda + \frac{\lambda}{C_2 |D|} \int_D N \left( \frac{|V(x)|}{\lambda} \right) dx; \lambda > 0 \right\} < \infty. \quad (2.11)$$

One can verify that  $\|\cdot\|_{N_D}$  defines a norm. For fixed  $V$ , we set  $F(\lambda) = \lambda \int_D N \left( \frac{V_+(x)}{\lambda} \right) dx$ , so that  $\|V_+\|_{N_D} = \inf \left\{ \lambda + \frac{F(\lambda)}{C_2 |D|} \right\}$ .

**Theorem 2.4** *Suppose that (1.1) has a nontrivial solution for  $V \in L \log L(D)$ , where  $D \subset \mathbf{R}^2$ . Then, for every  $\lambda > 0$ ,*

$$\lambda C_2 |D| + F(\lambda) \geq 4\pi. \quad (2.12)$$

*Equality can be attained in (2.12) when  $u_*$  is an extremal for (2.9) and  $V = V_+$  is as in (1.12).*

Theorem 1.4 follows immediately.

*Proof of Theorem 2.4.* Fix  $u, V$ . Let  $U = 4\pi \frac{|u(x)|^2}{\|\nabla u\|_2^2}$ . For fixed  $\lambda > 0$ , set  $v = \frac{V_+(x)}{\lambda}$ . We claim the following version of Young's inequality:

$$Uv \leq M(U) + N(v) \quad (2.13)$$

with equality if and only if  $v = e^U$ . To prove this, first consider the case that  $v \geq e^U \geq 1$ . Then  $N(v) \geq \bar{N} = \int_1^v \min \{U(s), \ln(s)\} ds$ . The rectangle  $[0, U] \times [0, v]$  can be partitioned into two disjoint regions, with areas  $M(U)$  and  $\bar{N}$ , so  $Uv = M(U) + \bar{N}$ . This proves the claim when  $v \geq e^U$ ; the rest is similar.

By Green's identity (Lemma 5.4), the definition of  $U$  and (2.13),

$$\begin{aligned} \|\nabla u\|_2^2 &= \int_D |u(x)|^2 V(x) dx \leq \int_D |u(x)|^2 V_+(x) dx \\ &= \frac{\|\nabla u\|_2^2}{4\pi} \int_D U(x) V_+(x) dx \\ &\leq \frac{\|\nabla u\|_2^2}{4\pi} \left( \lambda \int_D M(U(x)) dx + F(\lambda) \right). \end{aligned} \quad (2.14)$$

By (2.9),  $\int_D M(U)dx \leq C_2|D|$  and (2.12) follows.

We now show that equality is attained. Flucher proved in [Fl] that equality occurs in (2.9) for some  $u_* \in W_0^{1,2}(D)$ . We can assume that  $\|\nabla u_*\|_2 = 1$ . Let  $U = U_* = 4\pi|u_*(x)|^2$ , so  $\int_D M(U_*)dx = C_2|D|$ . By Lemma 5.3,  $-\Delta u_* = Vu_*$ , where  $V = \omega^{-1}e^{4\pi|u_*(x)|^2}$  and  $\omega = \int_D |u_*(x)|^2 e^{4\pi|u_*(x)|^2} dx$ . Set  $\lambda = \omega^{-1}$ , so  $e^{U_*} = V_+\lambda^{-1} = v$  with equality in (2.13) for all  $x$ . Direct calculation gives  $\int_D |U_*V_+|dx = 4\pi$ . So, by integrating (2.13)

$$4\pi = \lambda \int_D M(U_*(x)) dx + F(\lambda) = \lambda C_2|D| + F(\lambda).$$

Thus for these choices of  $u$ ,  $V$  and  $\lambda$ , (2.12) is an equality.  $\square$ .

### 3. Minimal support results for other elliptic equations

In this section we prove minimal support results for other well-known differential equations. Our linear examples are operators in divergence form and Schrödinger equations with first order terms. We also study some related non-linear elliptic equations.

#### 3.1 Operators in divergence form

Our next theorem generalizes Theorem 1.1 to operators in divergence form. Let  $a$ ,  $b$  be positive  $L^\infty(D)$  functions with  $\frac{1}{a}$ ,  $\frac{1}{b}$  in  $L^\infty(D)$ . Define the weighted space  $L^{p,b}(D)$  using the norm

$$\|u\|_{p,b}^p = \int_D |u(x)|^p b(x) dx$$

and define  $W_0^{1,2,a}(D)$  as the closure of  $C_0^\infty(D)$  with respect to  $\|\nabla u\|_{2,a}$ . These norms are equivalent to the ones with  $a \equiv b \equiv 1$ , and hence we have the usual compact embeddings  $W_0^{1,2,a}(D) \rightarrow L^{2q,b}(D)$  for  $q < \bar{q}$ , and for  $q = \infty$  when  $n = 1$ . When  $n > 2$  and  $q = \bar{q}$ , this embedding is bounded, but not compact. In what follows, we will denote by  $K = K(D, n, 2q, a, b)$  the best constant in the weighted Sobolev embedding theorem

$$\|u\|_{2q,b} \leq K \|\nabla u\|_{2,a}. \quad (3.1)$$

Let  $u \in W_0^{1,2,a}(D)$  be a non-trivial solution for

$$-\operatorname{div}(a\nabla u)(x) = V(x)b(x)u(x), \quad (3.2)$$

in the sense that

$$\int_D a\nabla u \cdot \nabla \psi dx = \int_D Vu\psi b dx \quad \forall \psi \in C_0^\infty(D).$$

**Theorem 3.1** Suppose  $V \in L^{r,b}(D)$  with  $r, q$  as in Theorem 1.1. Let  $u \in W_0^{1,2,a}(D)$  be a non-trivial solution of (3.2). Then

$$K^2 \|V\|_{r,b} \geq 1. \quad (3.3)$$

Equality can occur when  $r > \frac{n}{2}$  for  $n \geq 2$  and when  $n = 1$  and  $1 < r \leq \infty$ .

*Proof.* Let  $V \in L^{r,b}(D)$  with  $r \geq \frac{n}{2}$ . With Green's identity, (Lemma 5.4), we have the minimal support sequence

$$\|u\|_{2q,b}^2 \leq K^2 \int_D a |\nabla u|^2 dx = K^2 \int_D V |u|^2 b dx \quad (3.4)$$

$$\leq K^2 \|V\|_{r,b} \|u^2\|_{q,b} = K^2 \|V\|_{r,b} \|u\|_{2q,b}^2, \quad (3.5)$$

hence (3.3). Now let  $r > \frac{n}{2}$ . Since the weighted norms here are equivalent to the ones used in Lemmas 5.1 and 5.2, the proofs there still hold; there is a non-negative  $u_* \in W_0^{1,2,a}(D)$  for which  $\|u_*\|_{2q,b} = K \|\nabla u_*\|_{2,a}$ . It solves (3.2) with  $V = c |u_*(x)|^{2q-2}$ , where  $c = \frac{\|\nabla u_*\|_{2,a}^2}{\|u_*\|_{2q,b}^{2q}}$ . Note that (3.4) is an equation when  $u = u_*$ . Since  $\|u_*\|_{2q,b}^{2q} = \|u_*^2\|_{q,b}^q = \|u_*^{2q-2}\|_{r,b}^r$ , we get

$$\int_D V |u|^2 b dx = c \|u_*\|_{2q,b}^{2q} = c \|u_*^2\|_{q,b} \|u_*^{2q-2}\|_{r,b} = \|u_*^2\|_{q,b} \|V\|_{r,b}$$

so equality is also attained in Hölder's inequality (3.5), hence in (3.3) as well.  $\square$

### 3.2 A result for annuli

Theorem 1.1 can be extended to  $V \in L^r(D)$  for  $r > 1$  (instead of  $r > \frac{n}{2}$ ) in the special case of radial solutions of the Schrödinger equation (1.1) on an annulus in  $\mathbf{R}^n$ , with  $n \geq 2$ . Let  $\rho = |x|$ . Fix  $0 < c < d < \infty$ ; let  $I = (c, d)$  with the measure  $\rho^{n-1} d\rho$  and let  $A = \{x \in \mathbf{R}^n : c < \rho < d\}$ . Denote the set of radial functions in  $W_0^{1,2}(A)$  by  $W_{0,\text{rad}}^{1,2}(A)$  and define  $L_{\text{rad}}^p(A)$  similarly. If  $-\Delta u = Vu$  holds for  $u \in W_{0,\text{rad}}^{1,2}(A)$ , it follows that  $V = -\Delta u/u$  is also a radial function. The Laplacian of a radial function  $u$  is  $\Delta u = \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \left( \rho^{n-1} \frac{\partial}{\partial \rho} \right) u$ , so  $-\Delta u = Vu$  reduces to

$$-\frac{\partial}{\partial \rho} \left( \rho^{n-1} \frac{\partial}{\partial \rho} \right) u = \rho^{n-1} V(\rho) u \quad (3.6)$$

with  $u \in W_0^{1,2,\rho^{n-1}}(I)$ . Let  $\omega_n = |S^{n-1}|$ . Then the mapping  $u(|x|) \rightarrow \omega_n^{\frac{1}{p}} u(\rho)$  gives an isometric isomorphism  $L_{\text{rad}}^p(A) \rightarrow L^{p,\rho^{n-1}}(I)$ . Let

$$K_{q,\text{rad}}(A) = \sup_{u \in W_{0,\text{rad}}^{1,2}(A)} \frac{\|u\|_{2q}}{\|\nabla u\|_2} = \omega_n^{\frac{1}{2q}-\frac{1}{2}} \sup_{u \in W_0^{1,2,\rho^{n-1}}(I)} \frac{\|u\|_{2q,\rho^{n-1}}}{\|u'\|_{2,\rho^{n-1}}} = \omega_n^{\frac{1}{2q}-\frac{1}{2}} K \quad (3.7)$$

where  $K = K(I, 1, 2q, \rho^{n-1}, \rho^{n-1})$  is as in (3.1). We note that  $W_{0,\text{rad}}^{1,2}(A) \subset W_0^{1,2}(A)$  implies  $K_{q,\text{rad}}(A) \leq K_q(A)$ .

**Theorem 3.2** *Suppose the Schrödinger equation (1.1) has a nontrivial solution  $u \in W_{0,\text{rad}}^{1,2}(A)$  for some  $V \in L^r(A)$ , with  $1 < r \leq \infty$  and  $q = r^*$ . Then*

$$K_{q,\text{rad}}^2(A) \|V_+\|_r \geq 1 \quad (3.8)$$

and equality can be attained.

*Proof.* We apply Theorem 3.1 with  $D = I = (c, d)$ , and  $a(x) = b(x) = \rho^{n-1}$ . We obtain  $K^2 \|V_+\|_{r,\rho^{n-1}} \geq 1$ . By (3.7),  $K^2 = \omega_n^{1-\frac{1}{q}} K_{q,\text{rad}}^2(A)$ , and since

$$\omega_n^{1-\frac{1}{q}} \|V_+\|_{r,\rho^{n-1}} = \omega_n^{\frac{1}{r}} \|V_+\|_{r,\rho^{n-1}} = \|V_+\|_{L^r(A)},$$

(3.8) follows.

Let us show that equality can be attained; by Lemma 5.1, there exists  $u_* \in W_0^{1,2,\rho^{n-1}}(I)$  that satisfies  $\|u_*\|_{2q,\rho^{n-1}} = K \|u_*'\|_{2,\rho^{n-1}}$ . Define  $v_* \in W_{0,\text{rad}}^{1,2}(A)$  by  $v_*(x) = u_*(|x|)$ . We see at once that  $\|v_*\|_{2q} = K \omega_n^{\frac{1}{2q} - \frac{1}{2}} \|\nabla v_*\|_2 = K_{q,\text{rad}}(A) \|\nabla v_*\|_2$ . So,  $v_*$  is an extremal of (3.7). We apply Lemma 5.2 on  $I$ , with  $a = b = \rho^{n-1}$ , to get the following equation holding in the distribution sense on  $I$ :  $-\frac{\partial}{\partial \rho}(\rho^{n-1} \frac{\partial}{\partial \rho} u_*) = \rho^{n-1} \tilde{V} u_*$ , with

$$\tilde{V}(\rho) = \frac{|u_*(\rho)|^{2q-2} \|u_*'\|_{2,\rho^{n-1}}^2}{\|u_*\|_{2q,\rho^{n-1}}^{2q}} = \frac{|u_*(\rho)|^{2q-2} \|\nabla v_*\|_2^2}{\|v_*\|_{2q}^{2q}}.$$

Since  $v_*(x) = u_*(\rho)$ , by the discussion leading up to (3.6) we get  $-\Delta v_* = V v_*$ , with  $V(x) = |v_*(x)|^{2q-2} \|\nabla v_*\|_2^2 / \|v_*\|_{2q}^{2q}$ . Since  $(q-1)r = q$ ,

$$\left( \int_A |v_*(x)|^{(2q-2)r} dx \right)^{\frac{1}{r}} = \|v_*\|_{2q}^{2q-2}$$

and

$$K_{q,\text{rad}}^2(A) \|V\|_r = K_{q,\text{rad}}^2(A) \frac{\|\nabla v_*\|_2^2}{\|v_*\|_{2q}^{2q}} = 1. \quad \square$$

### 3.3 Minimal support results for $-\Delta u = Vu + W \cdot \nabla u$

In this section, we prove minimal support results for solutions of second order elliptic equations with first order terms. Specifically, we consider

$$-\Delta u = Vu + W \cdot \nabla u. \quad (3.9)$$

Throughout this section,  $W$  has values in  $\mathbf{R}^n$  and is defined on  $D \subset \mathbf{R}^n$ , with  $n \geq 1$ ;  $r$  and  $q = r^*$  are as in Theorem 1.1. The equation (3.9) is assumed to hold in the distribution sense, i.e.

$$\int_D \nabla u \cdot \nabla \psi dx = \int_D (Vu + W \cdot \nabla u) \psi dx, \quad \forall \psi \in C_0^\infty(D).$$

**Theorem 3.3** *Suppose that (3.9) has a nontrivial solution  $u \in W_0^{1,2}(D)$ , with  $V \in L^r(D)$  and  $W \in W^{1,r}(D, \mathbf{R}^n)$ ,  $r > \frac{n}{2}$ . Then,*

$$K_q^2(D) \left\| V - \frac{1}{2} \operatorname{div} W \right\|_r \geq 1 \quad (3.10)$$

and equality can be attained.

Equality in (3.10) can be attained when  $W \equiv 0$ , for then Theorem 3.3 reduces to Theorem 1.1. The theorem also holds, with the same proof, when  $r = \frac{n}{2}$  and  $n \geq 3$ . In this case, the inequality in (3.10) is strict, and it is sharp because Theorem 3.3 reduces to Theorem 2.1 when  $W \equiv 0$ .

*Proof.* By taking real or imaginary parts of (3.9) we can assume  $u$  real-valued. We assume  $n \geq 3$ ; the proofs for  $n = 1, 2$  are similar. By Sobolev's embedding theorem,  $W^{1,r}(D, \mathbf{R}^n) \subseteq L^n(D, \mathbf{R}^n)$ , because  $r \geq \frac{n}{2}$ . Since  $|\nabla u| \in L^2(D)$ , Hölder's inequality implies that  $\nabla u \cdot W \in L^p(D)$  with  $p = \frac{2n}{n+2} = (2\bar{q})^*$ , also that  $Vu \in L^p(D)$ . So we can apply Green's identity (Lemma 5.5, with  $a \equiv 1$  and  $F = Vu + W \cdot \nabla u$ ) to get

$$\|u\|_{2q}^2 \leq K_q^2(D) \|\nabla u\|_2^2 = K_q^2(D) \int_D (Vu + \nabla u \cdot W) u \, dx. \quad (3.11)$$

The same argument used to prove Lemma 5.4 justifies the identity

$$\int_D 2u \nabla u \cdot W \, dx = \int_D \nabla(u^2) \cdot W \, dx = - \int_D u^2 \operatorname{div} W \, dx. \quad (3.12)$$

From (3.11), (3.12) and Hölder's inequality it follows that

$$\|u\|_{2q}^2 \leq K_q^2(D) \int_D u^2 \left( V - \frac{1}{2} \operatorname{div} W \right) \, dx \leq K_q^2(D) \|u\|_{2q}^2 \left\| V - \frac{1}{2} \operatorname{div} W \right\|_r. \quad (3.13)$$

We conclude that  $K_q^2(D) \left\| V - \frac{1}{2} \operatorname{div} W \right\|_r \geq 1$ .  $\square$

In the next theorem, we prove a minimal support result for the solutions of (3.9) under weaker assumptions on  $W$ .

**Theorem 3.4** *Suppose that the differential equation (3.9) has a nontrivial solution  $u \in W_0^{1,2}(D)$  with  $V \in L^r(D)$  and  $W \in L^s(D, \mathbf{R}^n)$ . Let  $s \geq 2r \geq n$  (but if  $n = 2$ , let  $r > 1$ ). Then*

$$K_q(D) \left( K_q(D) \|V\|_r + |D|^{\frac{1}{2r} - \frac{1}{s}} \|W\|_s \right) \geq 1. \quad (3.14)$$

*Proof.* Again, we can assume that  $u$  is real-valued. Since  $s \geq n$ , the proof of (3.11) still holds, and gives a similar formula:

$$\|u\|_{2q} \|\nabla u\|_2 \leq K_q(D) \|\nabla u\|_2^2 = K_q(D) \left( \int_D u^2 V \, dx + \int_D u \nabla u \cdot W \, dx \right)$$

Applying Hölder's inequality with exponents  $q$  and  $r$  to the first integral, and Hölder's inequality with exponents  $2q$ ,  $2$ ,  $s$  and  $\left(\frac{1}{2} - \frac{1}{2q} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{2r} - \frac{1}{s}\right)^{-1}$  to the second integral (if  $s = 2r$ , the last exponent is not needed),

$$\|u\|_{2q} \|\nabla u\|_2 \leq K_q(D) \left( \|u\|_{2q}^2 \|V\|_r + \|u\|_{2q} \|\nabla u\|_2 \|W\|_s |D|^{\frac{1}{2r} - \frac{1}{s}} \right).$$

Applying Sobolev's inequality (1.2) to the first summand on the right hand side,

$$\|u\|_{2q} \|\nabla u\|_2 \leq K_q(D) \|u\|_{2q} \|\nabla u\|_2 \left( K_q(D) \|V\|_r + |D|^{\frac{1}{2r} - \frac{1}{s}} \|W\|_s \right).$$

So,

$$1 \leq K_q(D) \left( K_q(D) \|V\|_r + |D|^{\frac{1}{2r} - \frac{1}{s}} \|W\|_s \right). \quad \square$$

We conclude with a corollary to Theorem 3.1.

**Theorem 3.5** *Suppose that (3.9) has a nontrivial solution  $u \in W_0^{1,2}(D)$  for  $V \in L^r(D)$ , with  $r > \frac{n}{2}$ , and for  $W \in L^\infty(D, \mathbf{R}^n)$ . Suppose also that  $W$  is exact, i.e.  $W = \nabla\phi$  for some  $\phi \in W^{1,\infty}(D)$ . Let*

$$K_{q,\phi}(D) = \sup_{u \neq 0} \frac{\|u\|_{2q,e^\phi}}{\|\nabla u\|_{2,e^\phi}}.$$

Then

$$K_{q,\phi}^2(D) \|V_+\|_{r,e^\phi} \geq 1 \tag{3.15}$$

and equality can be attained.

*Proof.* Since  $-\Delta u - W \cdot \nabla u = Vu$  and  $W = \nabla\phi$ , we have  $-\operatorname{div}(e^\phi \nabla u) = e^\phi Vu$ . Applying Theorem 3.1 with  $a = b = e^\phi$  immediately yields the desired result.  $\square$

Theorem 3.5 has the advantage of being sharp for any exact  $W$ , but the estimate in Theorem 3.3 has the advantage that it does not involve the weight  $e^\phi$ .

### 3.4 Some nonlinear equations

In this section we study certain nonlinear differential equations. We start with the equation

$$-\Delta u = V|u|^{\beta-1}u \tag{3.16}$$

where  $1 \leq \beta$ . Here  $V$  is assumed to be real, but  $u$  can be complex. We say that  $u \in W_0^{1,2}(D)$  is a *very weak solution* of the equation (3.16) if

$$\int_D \nabla u \cdot \nabla \psi \, dx = \int_D V|u|^{\beta-1}u\psi \, dx, \quad \forall \psi \in C_0^\infty(D).$$

**Theorem 3.6** Assume that (3.16) has a nontrivial very weak solution  $u \in W_0^{1,2}(D)$ . Let  $\hat{q} = q(\beta + 1)/2$  and assume  $\hat{q} \leq \bar{q}$ . If  $n \leq 2$ , let  $\hat{q} < \infty$ . If  $V \in L^r(D)$  with  $r = q^*$ , then

$$K_{\hat{q}}^2(D) \|V_+\|_r \|u\|_{q(\beta+1)}^{\beta-1} \geq 1. \quad (3.17)$$

Equality can be attained in (3.17) when  $\hat{q} < \bar{q}$ .

*Proof.* Assume  $n \geq 3$ ; the proof is similar for  $n \leq 2$ . By Sobolev's inequality,  $u \in L^{2\bar{q}}(D)$ . A calculation shows that  $V|u|^{\beta-1} \in L^{\bar{q}^*}(D)$ , allowing Green's identity (Lemma 5.4A with  $V|u|^{\beta-1}$  replacing  $V$ ) in the minimal support sequence below.

$$\begin{aligned} \|u\|_{q(\beta+1)}^2 &= \|u\|_{2\hat{q}}^2 \leq K_{\hat{q}}^2 \|\nabla u\|_2^2 = K_{\hat{q}}^2 \int_D |u(x)|^{\beta+1} V(x) dx \\ &\leq K_{\hat{q}}^2 \int_D |u(x)|^{\beta+1} V_+(x) dx \leq K_{\hat{q}}^2 \|u\|_{(\beta+1)q}^{\beta+1} \|V_+\|_r \end{aligned} \quad (3.18)$$

from which (3.17) follows. If  $\hat{q} < \bar{q}$ , Lemmas 5.1 and 5.2 provide a  $u_* \geq 0$  such that  $-\Delta u_* = cu_*^{2\hat{q}-1}$  with

$$c = \frac{\|\nabla u_*\|_2^2}{\|u_*\|_{2\hat{q}}^{2\hat{q}}} = K_{\hat{q}}^{-2} \|u_*\|_{2\hat{q}}^{2-2\hat{q}}.$$

So,  $-\Delta u_* = Vu_*^\beta$ , which is (3.16), with  $V = V_+ = cu_*^{2\hat{q}-1-\beta} = cu_*^{(q-1)(\beta+1)}$ . Thus,  $\|V_+\|_r = c\|u_*\|_{q(\beta+1)}^{(q-1)(\beta+1)} = c\|u_*\|_{2\hat{q}}^{2\hat{q}-\beta-1}$ , which gives equality in (3.17).  $\square$

We now consider the equation

$$-\Delta u = V|\nabla u|^\beta \quad (3.19)$$

with  $0 < \beta \leq 2$ . The case  $\beta = 2$  is particularly interesting and well studied in the literature, (see e.g. [C] and the references cited there). We assume that  $u \in W_0^{1,2}(D)$  is a nontrivial weak solution of (3.19), in the sense that

$$\int_D \nabla u \cdot \nabla \psi dx = \int_D |\nabla u|^\beta V \psi dx \quad (3.20)$$

for every  $\psi \in W_0^{1,2}(D)$ . In the following theorem, we depart from our convention that  $q = r^*$ .

**Theorem 3.7** Let  $u$  be as in (3.19) with  $0 < \beta < 2$ ,  $V \in L^r(D)$ ,  $q < \bar{q}$  and  $\frac{1}{2q} + \frac{\beta}{2} + \frac{1}{r} = 1$ . Then

$$K_q(D) \|\nabla u\|_2^{\beta-1} \|V\|_r \geq 1. \quad (3.21)$$

When  $\beta = 2$ ,  $\|Vu\|_\infty \geq 1$ .

*Proof.* By Sobolev's inequality and (3.20), with  $\psi = \bar{u}$ , and by Hölder's inequality with exponents  $\frac{2}{\beta}$ ,  $2q$ , and  $r$  we have the following minimal support sequence

$$\|u\|_{2q} \|\nabla u\|_2 \leq K_q(D) \|\nabla u\|_2^2 = K_q(D) \int_D \bar{u} V |\nabla u|^\beta dx \leq K_q(D) \|u\|_{2q} \|\nabla u\|_2^\beta \|V\|_r,$$



which implies (3.21). When  $\beta = 2$ , (3.20) shows  $\int_D |\nabla u|^2 (1 - V\bar{u}) dx = 0$ . If  $\|Vu\|_\infty < 1$ , then  $|1 - V\bar{u}| > 0$  a.e., so  $\nabla u \equiv 0$  on  $D$  and  $u \equiv 0$ , a contradiction.  $\square$

When  $n \neq 2$ , (3.21) is also valid for  $q = \bar{q}$ .

## 4. Minimal support results in one dimension

In this section, we let  $n = 1$  and  $D = (-b, b)$ . We consider nontrivial solutions in the distribution sense of the equation

$$-u''(x) = V(x)u(x) \quad u \in W_0^{1,2}(D). \quad (4.1)$$

We can assume without loss of generality that  $u$  is real-valued. We can extend  $u$  continuously to  $[-b, b]$  by setting  $u(-b) = u(b) = 0$ . As noted elsewhere, most of the results in this paper hold in this setting, but in the next section we show how the case  $r = 1$  differs; the extremal  $V$  is a measure, rather than a function. In Section 4.2 we compute sharp constants in Sobolev inequalities and minimal support results on  $\mathbf{R}^1$ .

### 4.1 Potentials in $L^1$

When  $n = 1$ , we can consider  $V \in L^1(D)$  in Theorem 1.1, so that  $q = \infty$ . The minimal support sequence still holds in this case, but the variational work in Lemma 5.2 does not, so interesting new questions on sharpness and extremals arise. We prove an analogue of Theorem 1.1, replacing  $L^1(D)$  with the space  $M$  of signed measures  $V$  on  $D$ , (see e.g. [Ru] for the definition and properties of signed measures) with norm

$$\|V\|_M = |V|(D) < \infty$$

In the special case where  $V \in L^1(D)$ , we have  $\|V\|_M = |V|(D) = \int_D |V(x)| dx = \|V\|_1$ .

**Theorem 4.1** *Assume  $u \in W_0^{1,2}(-b, b)$  is a nontrivial solution of (4.1), with  $V \in M$ . Then*

$$b\|V\|_M \geq 2. \quad (4.2)$$

*Equality is attained when  $u = 1 - \frac{|x|}{b}$ . Equality is not possible with  $V \in L^1(-b, b)$ , but (4.2) is still sharp in this case.*

*Remark:* For  $w \in W^{1,1}(-b, b)$  we have (see [E], p.286):

$$w(t) - w(s) = \int_s^t w'(\tau) d\tau. \quad (4.3)$$

We can apply this with  $w = u$  and also with  $w = u'$ ; since  $V \in L^1(-b, b)$ , and  $u \in L^\infty(-b, b)$ , (4.1) implies  $u'' \in L^1(-b, b)$ . We will use (4.3) without further comment throughout this section.

*Proof.* For  $x \in (-b, b)$ , by (4.3) and Hölder's inequality

$$|u(x)| \leq \int_{-b}^x |u'| dt \leq \left( (x+b) \int_{-b}^x |u'|^2 dt \right)^{\frac{1}{2}}.$$

Likewise,

$$|u(x)| \leq \left( (b-x) \int_x^b |u'|^2 dt \right)^{\frac{1}{2}}.$$

By squaring and algebra,

$$\frac{2|u(x)|^2}{b} \leq \left( \frac{1}{x+b} + \frac{1}{b-x} \right) |u(x)|^2 \leq \|u'\|_2^2$$

So,  $K_\infty^2(-b, b) \leq \frac{b}{2}$ . Now, apply the minimal support sequence for equation (4.1), with  $q = \infty$ . The "Hölder step" in the sequence can be replaced by

$$\int u^2 dV \leq \|u^2\|_\infty \|V\|_M$$

and we get (4.2).

Setting  $b = 1$  for simplicity, the claim about  $u = 1 - |x|$  it is easy to verify directly. For this  $u$ ,  $\frac{V}{2}$  is a Dirac mass at  $x = 0$ . We see that (4.2) is sharp for  $V \in L^1(-1, 1)$  by considering an approximating sequence to  $u = 1 - |x|$ . This also implies that  $K_\infty(-b, b) = \sqrt{\frac{b}{2}}$ .

Now, we prove that for  $V \in L^1(-b, b)$  equality is never attained, (this reasoning also gives an independent proof of (4.2) for this case). We may assume  $u > 0$  on  $(-b, b)$ , for if it changes sign, we may apply (4.2) to a restriction of  $u$ , and we are done. For now, suppose that  $u$  attains its maximum value at a unique point  $c \in (-b, b)$ . As in the remark above,  $u'$  is defined and continuous on  $(-b, b)$ , and so  $u'(c) = 0$ . Next, we claim that

$$\int_c^b |V| dx > \frac{1}{b-c}. \tag{4.4}$$

To prove this, we may assume that  $u(c) = 1$ . By the mean value theorem on  $[c, b]$ , there is a point  $c < d < b$  such that  $u'(d) = -\frac{1}{b-c}$ . Since  $u$  is maximal only at  $c$ , we have  $0 < u < 1$  and  $\frac{1}{u} > 1$  on  $(c, b)$ , and

$$\int_c^d |V| dx = \int_c^d \frac{|u''|}{u} dx > \int_c^d |u''| dx = \frac{1}{b-c}.$$

The claim follows. Similarly,  $(b+c) \int_{-b}^c |V| dx > 1$ . So,

$$\|V\|_M = \|V\|_1 > \frac{1}{b-c} + \frac{1}{b+c} \geq \frac{2}{b},$$

proving that (4.2) is strict. We have assumed that  $u$  attains a maximum only at one point  $c$ ; the general case follows by similar reasoning applied to appropriate restrictions of  $u$ .  $\square$

## 4.2 Sobolev constants and extremals

Some extremals and sharp constants can be computed explicitly when  $n = 1$ . Sharp Sobolev constants for the embedding  $W^{1,p}(D) \rightarrow L^p(D)$  on an interval  $D$  are found in [TMH]. When  $r = \infty$  and  $D = (-1, 1)$ , an explicit extremal for (1.3) in Theorem 1.1 is  $u(x) = \cos(\frac{\pi x}{2})$ , with  $V(x) = (\frac{\pi}{2})^2$ , a constant. For this  $V$ ,  $K_1^2(D) \|V\|_\infty = 1$ , which proves the well-known formula for the sharp Sobolev constant,  $K_1(D) = |D| \pi^{-1}$ . Also, in Section 4.1 we used the explicit extremal  $u = 1 - |x|$  for (1.3) to prove that  $K_\infty^2(D) = \frac{|D|}{4}$ .

Next we study the Sobolev constants  $K_q(D)$ ,  $1 < q < \infty$ . We remark without proof that the corresponding extremals extend smoothly to periodic functions with  $-u'' = |u|^{2q-2}u$  on  $\mathbf{R}$ , but we will not use this. Recall that Theorem 1.6 asserts that  $K_q = b^{\frac{1}{2q} + \frac{1}{2}} M_q$ , where

$$M_q = \frac{2\sqrt{q} \Gamma\left(\frac{q+1}{2q}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{2q}\right)} \left(\frac{q+1}{2}\right)^{\frac{q-1}{2q}}. \quad (4.5)$$

*Proof of Theorem 1.6.* Let  $u = u_*$  be an extremal for the the Sobolev inequality (1.2). We may assume  $u > 0$  on  $(-b, b)$  with  $u(-b) = u(b) = 0$ . By spherical symmetrization (see [LL]), we may assume  $u$  is even, that  $u$  achieves its maximum at  $x = 0$ , and that  $u'(x) \leq 0$  on  $(0, b)$ . By Lemma 5.2,  $-u'' = cu^{2q-1} = Vu$ , where  $c = \|u'\|_2^2 \|u\|_{2q}^{-2q}$ .

By Theorem 1.1,  $K_q^2 \|V\|_r = 1$ . By dilation, we can assume that  $\|V\|_r = 1$ , so that  $K_q = 1$  for the resulting domain, denoted by  $D_q = [-b_q, b_q]$ . For general intervals, by dilation,

$$K_q(D) = \left(\frac{|D|}{|D_q|}\right)^{\frac{1}{2q} + \frac{1}{2}} = \left(\frac{b}{b_q}\right)^{\frac{1}{2q} + \frac{1}{2}}.$$

It remains to compute  $b_q$  and show

$$b_q^{-\frac{q+1}{2q}} = M_q. \quad (4.6)$$

Normalizing, we can assume  $\|u\|_{2q} = 1$ . Since  $K_q = 1$ , this implies  $\|u'\|_2 = 1$ ,  $c = 1$  and  $-u'' = u^{2q-1}$ . This shows  $2qu'u'' + 2qu^{2q-1}u' = 0$ . Integrating, we get

$$q(u')^2 + u^{2q} = u^{2q}(0). \quad (4.7)$$

Integrating (4.7), we get  $q + 1 = 2b_q u^{2q}(0)$ . Since  $u'' < 0$ , (4.3) implies  $u' < 0$  on  $(0, b_q)$  and  $u$  decreases there. So,  $u$  restricted to  $[0, b_q]$  has an inverse function  $v$  defined on  $[0, u(0)]$ . From calculus,  $v'(y) = \frac{1}{u'(x)}$  and from (4.7), this equals  $\sqrt{q}(u^{2q}(0) - u^{2q}(x))^{-\frac{1}{2}}$  and with  $u = y$ . So,

$$b_q = - \int_0^{u(0)} v'(y) dy = - \int_0^{u(0)} \frac{1}{u'} dy = q^{\frac{1}{2}} \int_0^{u(0)} (u^{2q}(0) - y^{2q})^{-\frac{1}{2}} dy.$$

Now substitute  $y = u(0)z$ , to get

$$\int_0^{u(0)} (u^{2q}(0) - y^{2q})^{-\frac{1}{2}} dy = u^{1-q}(0) \int_0^1 (1 - z^{2q})^{-\frac{1}{2}} dz = u^{1-q}(0) I.$$

Recall  $q + 1 = 2b_q u^{2q}(0)$ , so

$$u^{1-q}(0) = \left( \frac{q+1}{2b_q} \right)^{\frac{1-q}{2q}}.$$

Combining,  $b_q = q^{\frac{1}{2}} \left( \frac{q+1}{2b_q} \right)^{\frac{1-q}{2q}} I$ , or

$$b_q^{\frac{1+q}{2q}} = \sqrt{q} \left( \frac{1+q}{2} \right)^{\frac{1-q}{2q}} I.$$

With the substitution  $z^{2q} = \sin^2 \theta$ ,  $I = \frac{1}{q} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{1}{q}-1} d\theta$ . Recalling the well-known formula  $\int_0^{\frac{\pi}{2}} (\sin x)^a (\cos x)^b dx = \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}{2\Gamma(\frac{1}{2}(a+b+2))}$ , which is valid for  $a, b > -1$ , we can see at once that  $I = \frac{\sqrt{\pi}\Gamma(\frac{1}{2q})}{2q\Gamma(\frac{q+1}{2q})}$  from which (4.5) and (4.6) follow.  $\square$

## 5. Appendix

In this section, we prove various lemmas needed throughout the paper. Some already appear in the literature in slightly different form, but are presented here for completeness. We establish existence of extremals, some variational formulas, and several versions of Green's identity.

### 5.1 Existence of Sobolev extremals

We first prove the existence of extremals for the weighted Sobolev inequality (3.1) used in Theorem 3.1. This applies in other settings, such as Theorem 1.1, when the weights are  $a = b = 1$ . See [E] for the functional analysis used in the lemmas below.

**Lemma 5.1** *Let  $q < \bar{q}$  for  $n \geq 2$ , and  $1 \leq q \leq \infty$  for  $n = 1$ . Let  $a, b \in L^\infty(D)$ , with  $\frac{1}{a}, \frac{1}{b} \in L^\infty(D)$ ; define  $K$  as in (3.1). There is a nontrivial and non-negative  $u_* \in W_0^{1,2,a}(D)$  for which*

$$\|\nabla u_*\|_{2,a} = K \|u_*\|_{2q,b}. \quad (5.1)$$

*Proof.* Let  $B_W$  denote the set of all elements of  $W_0^{1,2,a}(D)$  with  $\|\nabla u\|_{2,a} \leq 1$ . By our assumptions on the weights  $a$  and  $b$ , the norms in  $L^{q,b}(D)$  and  $W_0^{1,2,a}(D)$  are equivalent to the norms with  $a \equiv b \equiv 1$ . Thus,  $B_W$  is weakly compact in  $W_0^{1,2,a}(D)$ . By the Kondrachov-Rellich Theorem for  $n \geq 2$ , and by the Arzela-Ascoli theorem for  $n = 1$ , the inclusion  $W_0^{1,2,a}(D) \rightarrow L^{2q,b}(D)$  is compact. Let  $\{u_n\}$  be a sequence in  $W_0^{1,2,a}(D)$  such that

$$\lim_{n \rightarrow \infty} \frac{\|u_n\|_{2q,b}}{\|\nabla u_n\|_{2,a}} = K.$$

We can assume by scaling that  $\|\nabla u_n\|_{2,a} = 1$ . Since  $B_W$  is weakly compact in  $W_0^{1,2,a}(D)$ , there exists a subsequence  $\{u_{n_k}\} \subset \{u_n\}$  that converges weakly to some  $u_* \in B_W$ . By the compactness of the inclusion  $W_0^{1,2,a}(D) \rightarrow L^{2q,b}(D)$ , there is a subsequence of  $\{u_{n_k}\}$ , that we label again with  $\{u_{n_k}\}$ , that converges to some  $w \in L^{2q,b}(D)$  in the strong topology of  $L^{2q,b}(D)$ . That is,  $\lim_{k \rightarrow \infty} \|u_{n_k} - w\|_{2q,b} = 0$ . But  $u_{n_k} \rightarrow w$  also in the weak topology of  $L^{2q,b}(D)$ , and so  $u_* = w$  a.e.; consequently,  $w \in B_W$  and  $\|\nabla w\|_{2,a} \leq 1$ . We have

$$K = \lim_{k \rightarrow \infty} \|u_{n_k}\|_{2q,b} = \|w\|_{2q,b}, \quad (5.2)$$

and so  $\frac{\|w\|_{2q,b}}{\|\nabla w\|_{2,a}} \geq K$ . But recall that  $w = u_* \in W_0^{1,2,a}(D)$ , and so  $\frac{\|w\|_{2q,b}}{\|\nabla w\|_{2,a}} \leq K$ , thus proving (5.1). We can replace  $u_*$  by  $|u_*|$ , if necessary, to get a non-negative extremal, with no effect on (5.1) (see [LL]).  $\square$

## 5.2 Variational work: divergence and Orlicz forms

We now show that the extremals of the Sobolev inequality (3.1) solve an equation of the form (3.2), and we give an explicit expression for  $V$  in terms of these extremals.

**Lemma 5.2** *Let  $q < \bar{q}$  with  $a, b$  and  $u_* \geq 0$  as in Lemma 5.1. Then,  $-\operatorname{div}(a\nabla u_*) = bVu_*$ , in the distribution sense, with*

$$V(x) = \frac{u_*(x)^{2q-2} \|\nabla u_*\|_{2,a}^2}{\|u_*\|_{2q,b}^{2q}}. \quad (5.3)$$

*Proof.* Let  $v(x) = u_*(x) + \epsilon\phi(x)$  where  $\phi \in C_0^\infty(D)$  is real-valued. Let  $\delta = \frac{d}{d\epsilon}|_{\epsilon=0}$ . Since  $u_*$  is an extremal for (3.1),  $\delta \left( \frac{\|u_* + \epsilon\phi\|_{2q,b}^2}{\|\nabla(u_* + \epsilon\phi)\|_{2,a}^2} \right) = 0$ . Direct computation yields

$$\begin{aligned} \delta \|u_* + \epsilon\phi\|_{2q,b}^2 &= \frac{1}{q} \|u_*\|_{2q,b}^{2-2q} \int_D \delta(b|u_* + \epsilon\phi|^{2q}) dx \\ &= 2 \|u_*\|_{2q,b}^{2-2q} \int_D u_*^{2q-1} \phi b dx \end{aligned}$$

and

$$\begin{aligned}\delta \|\nabla(u_* + \epsilon\phi)\|_{2,a}^2 &= \int_D \delta(a|\nabla(u_* + \epsilon\phi)|^2) dx \\ &= 2 \int_D (\nabla u_* \cdot \nabla \phi) a \, dx.\end{aligned}$$

One can justify passing the derivatives into the integrals by arguing as in [St]. By the quotient rule

$$\delta \left( \frac{\|u_* + \epsilon\phi\|_{2q,b}^2}{\|\nabla(u_* + \epsilon\phi)\|_{2,a}^2} \right) = 2 \frac{\|u_*\|_{2q,b}^2}{\|\nabla u_*\|_{2,a}^4} \int_D \left( \frac{\|\nabla u_*\|_{2,a}^2}{\|u_*\|_{2q,b}^{2q}} u_*^{2q-1} b \phi - a(\nabla u_* \cdot \nabla \phi) \right) dx = 0.$$

So, for every  $\phi \in C_0^\infty(D)$ , and for  $V$  as in (5.3)

$$\int_D V u_* b \phi \, dx = \int_D a(\nabla u_* \cdot \nabla \phi) dx$$

and  $-\operatorname{div}(a\nabla u_*) = bVu_*$ , as required.  $\square$

The next lemma shows that the extremals for the Moser-Trudinger inequality in  $\mathbf{R}^2$  also satisfy a Schrödinger equation.

**Lemma 5.3** *Let  $u_* \in W_0^{1,2}(D)$  be a extremal of the Moser-Trudinger inequality (2.9), with  $\|\nabla u_*\|_2 = 1$ ; then  $-\Delta u_* = Vu_*$  in the distribution sense, where*

$$V = \frac{e^{4\pi|u_*(x)|^2}}{\int_D |u_*|^2 e^{4\pi|u_*|^2} dx}. \quad (5.4)$$

*Proof.* As in Lemma 5.1, we can assume without loss of generality that  $u_*$  is non-negative. Let

$$U_\epsilon(x) = \frac{4\pi|u_*(x) + \epsilon\phi(x)|^2}{\|\nabla(u_* + \epsilon\phi)\|_2^2}$$

where  $\phi \in C_0^\infty(D)$  is real-valued. Also set  $U = U_0$ . Let  $\delta = \frac{d}{d\epsilon}|_{\epsilon=0}$ . Let  $M$  be as in the proof of Theorem 2.4. Then

$$\delta(M(U_\epsilon)) = \int_D e^{U_\epsilon} \delta(U_\epsilon) dx = 0 \quad (5.5)$$

and

$$\delta(U_\epsilon) = 8\pi \left( u_* \phi - |u_*|^2 \int_D \nabla u_* \cdot \nabla \phi \, dx \right).$$

One can justify the passing the derivatives into the integrals by arguing as in [St]. With  $\omega = \int_D |u_*|^2 e^{4\pi|u_*|^2}$ , we can simplify the equation (5.5) as follows:

$$\begin{aligned}0 &= \int_D e^U \left( u_* \phi - |u_*|^2 \left( \int_D \nabla u_* \cdot \nabla \phi \, dx \right) \right) dx \\ &= \int_D e^U u_* \phi \, dx - \omega \int_D \nabla u_* \cdot \nabla \phi \, dx = \omega \int_D (\omega^{-1} e^U u_* \phi - \nabla u_* \cdot \nabla \phi) \, dx.\end{aligned}$$

which proves (5.4).  $\square$

### 5.3 Green's identities for divergence and Orlicz forms.

The following lemmas substitute for Green's identity throughout the paper, often with  $a \equiv b \equiv 1$ .

**Lemma 5.4** *Let  $a(x) > 0$  with  $a, \frac{1}{a} \in L^\infty(D)$  and let  $b \in L^\infty(D)$ . Let  $u \in W_0^{1,2,a}(D)$ , with  $-\operatorname{div}(a\nabla u) = bVu$  in the distribution sense. Then the identity*

$$\int_D |\nabla u|^2 a dx = \int_D V|u|^2 b dx. \quad (5.6)$$

holds whenever either A), B) or C) hold:

- A)  $V \in L^{q^*}(D)$  when  $n \neq 2$ , and  $V \in L^r(D)$  for some  $r > 1$  when  $n = 2$ ,
- B)  $n \geq 3$  and either  $|V(x)| \leq c|x|^{-2}$  or  $|V(x)| \leq c(\operatorname{dist}(x))^{-2}$  (see (2.6)),
- C)  $u$  is real-valued,  $a \equiv b \equiv 1$ ,  $n = 2$  and  $V \in L \log L(D)$ .

*Proof.* By definition of solution in the distribution sense

$$\int_D \nabla u \cdot \nabla \psi a dx = \int_D Vu\psi b dx \quad (5.7)$$

for every  $\psi \in C_0^\infty(D)$ . The norm in  $W_0^{1,2,a}(D)$  is equivalent to the norm in  $W_0^{1,2}(D)$ , so  $C_0^\infty(D)$  is dense in both. Let  $\{\psi_n\}$  be a sequence of functions in  $C_0^\infty(D)$  that converges to  $\bar{u}$  in  $W_0^{1,2,a}(D)$ . Then

$$\int_D \nabla u \cdot \nabla \psi_n a dx - \int_D |\nabla u|^2 a dx \leq \|\nabla \psi_n - \nabla \bar{u}\|_{2,a} \|\nabla u\|_{2,a} \rightarrow 0.$$

To complete the proof of (5.6), it suffices to show that  $Vu\psi_n$  converges to  $V|u|^2$  in  $L^{1,b}(D)$  when  $V$  is as in A), B) or C). Assume A), and that  $n \neq 2$  (the proof is similar when  $n = 2$ ). By Sobolev's inequality,  $\psi_n$  converges to  $\bar{u}$  in  $L^{2q}(D)$ . By Holder's inequality,

$$\int_D |Vu\psi_n - V|u|^2| b dx \leq \|b\|_\infty \|V\|_{q^*} \|u\|_{2q} \|\psi_n - \bar{u}\|_{2q} \rightarrow 0.$$

For case B), first assume  $|V(x)| \leq c|x|^{-2}$ . Note that  $Vu\psi \in L^{1,b}(D)$  because  $|V| \leq C|x|^{-2}$ , and by the Hardy-Sobolev inequality (2.7),  $\psi|x|^{-1}$  and  $u|x|^{-1}$  are in  $L^2(D)$ . Let  $\{\psi_n\} \in C_0^\infty(D)$  be a sequence of functions that converges to  $\bar{u}$  in  $W_0^{1,2,a}(D)$ . We show that  $Vu\psi_n$  converges in  $L^{1,b}(D)$  to  $V|u|^2$ . By Hölder's inequality and (2.7)

$$\begin{aligned} & \int_D |Vu\psi_n - V|u|^2| b dx \leq \int_D c|x|^{-2}|u| |\psi_n - \bar{u}| b dx \\ & \leq c\|b\|_\infty \left( \int_D |x|^{-2}|u|^2 dx \right)^{\frac{1}{2}} \left( \int_D |x|^{-2} |\psi_n - \bar{u}|^2 dx \right)^{\frac{1}{2}} \leq c\|b\|_\infty \|\nabla(\psi_n - \bar{u})\|_2 \|\nabla u\|_2 \rightarrow 0. \end{aligned}$$

This proves (5.6) in this case. The proof is similar when  $V(x) \leq c(\text{dist}(x))^{-2}$ , using the inequality (2.8) instead of (2.7).

Next, we assume C), and without loss of generality, that  $\|\nabla u\|_2 = 1$ . Let  $\{\psi_n\} \subset C_0^\infty(D)$  converge to  $u$  in  $W_0^{1,2}(D)$ . We can choose  $\lambda_n \downarrow 0$  so that  $\|\nabla(u - \psi_n)\|_2 \lambda_n^{-1} \rightarrow 0$ . The Moser-Trudinger inequality implies

$$\int_D e^{\left(\frac{u-\psi_n}{\lambda_n}\right)^2} dx \leq \int_D e^{4\pi\left(\frac{u-\psi_n}{\lambda_n}\right)^2} dx \leq \int_D e^{4\pi\left(\frac{u-\psi_n}{\|\nabla(u-\psi_n)\|_2}\right)^2} dx < (C_2 + 1)|D| < \infty,$$

with  $C_2$  independent of  $u$  and  $\psi_n$ . A similar inequality holds when  $\frac{u-\psi_n}{\lambda_n}$  is replaced by  $u$ . Define the functions  $M_2(t) = e^t - t - 1$  and  $N_2(t) = (t + 1) \log(t + 1) - t$  for  $t \geq 0$ . These are complementary Orlicz functions, with properties similar to  $M$  and  $N$ , such as Young's inequality (2.13) (see also [KR]). Using this, the inequality  $2|ab| \leq a^2 + b^2$ , and Hölder's inequality:

$$\begin{aligned} \int_D |u V(u - \psi_n)| dx &= \lambda_n \int_D \left| \frac{u(u - \psi_n)}{\lambda_n} V \right| dx \\ &\leq \lambda_n \int_D M_2\left(\left|\frac{u(u - \psi_n)}{\lambda_n}\right|\right) dx + \lambda_n \int_D N_2(|V|) dx \\ &\leq \lambda_n \int_D e^{\left|\frac{u(u - \psi_n)}{\lambda_n}\right|} dx + \lambda_n \int_D N_2(|V|) dx \\ &\leq \lambda_n \int_D e^{\frac{u^2}{2}} e^{\frac{((u - \psi_n)\lambda_n^{-1})^2}{2}} dx + \lambda_n \int_D N_2(|V|) dx \\ &\leq \lambda_n \left\{ \left( \int_D e^{u^2} dx \right)^{\frac{1}{2}} \left( \int_D e^{\left(\frac{u - \psi_n}{\lambda_n}\right)^2} dx \right)^{\frac{1}{2}} + \int_D N_2(|V|) dx \right\}. \end{aligned}$$

Thus  $uV\psi_n$  converges in  $L^1(D)$  to  $Vu^2$ .  $\square$

The next lemma is used in Section 3. It contains Lemma 5.4 part A as a special case.

**Lemma 5.5** *Let  $a(x) > 0$  with  $a, \frac{1}{a} \in L^\infty(D)$ . Let  $u \in W_0^{1,2,a}(D)$  be a solution in the distribution sense of  $-\text{div}(a\nabla u) = F$ , where  $F \in L^{(2\bar{q})^*}(D)$  for  $n \neq 2$ , and  $F \in L^r(D)$  for some  $r > 1$  for  $n = 2$ . Then,*

$$\int_D |\nabla u|^2 a dx = \int_D F \bar{u} dx. \quad (5.8)$$

*Proof.* We have  $\int_D a\nabla u \cdot \nabla \psi dx = \int_D F \psi dx$  for every  $\psi \in C_0^\infty(D)$ . The rest is similar to the proof of Lemma 5.4, part A.

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