

Ingham-type inequalities for complex frequencies and applications to control theory

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December 20, 2005

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Abstract

For complex valued sequences $\{\omega_n\}_{n=1}^\infty$ of the form $\omega_n = a_n + ib_n$ with $a_n \in \mathbf{R}$ and $b_n \geq 0$, we prove inequalities of the form $\int_0^T |\sum_{n=p}^\infty x_n e^{it\omega_n}|^2 dt \asymp \sum_{n=p}^\infty |x_n|^2 / (1 + b_n)$, for all sequences $\{x_n\}$ with $\sum_{n=1}^\infty |x_n|^2 / (1 + b_n) < \infty$. We apply these to prove exact null-controllability for a class of hinged beam equations with mild internal damping with either boundary control or internal control.

1 Introduction

Let $a_1 < a_2 < \dots$ be a real valued sequence satisfying the gap condition: there exists a constant $\gamma > 0$ such that

$$|a_n - a_k| \geq \gamma|n - k|, \quad \forall n, k.$$

Then a theorem due to Ingham [8] states that for all $\epsilon > 0$, there exist positive constants C_1, C_2 such that

$$C_1 \sum_{n=1}^\infty |x_n|^2 \geq \int_0^{2\pi/\gamma+\epsilon} |\sum_{n=1}^\infty x_n e^{ita_n}|^2 dt \geq C_2 \sum_{n=1}^\infty |x_n|^2, \quad \forall \{x_n\} \in \ell^2.$$

These inequalities were originally used by Ingham to prove certain estimates on Dirichlet series. Later they were extended to complex valued sequences by various authors to study linear algebraic properties of families of exponential functions, see [19],[1] and references therein. More recently, various extensions have been used to prove controllability properties of systems of partial differential equations, see [10],[4],[2],[6] and references therein. In several of these works ([19],[2],[6],[1] and references therein), the sequence $\{a_n\}$ was replaced by a complex valued sequence $\{\omega_n\}$ with $\omega_n = a_n + ib_n$, but it was always assumed that b_n was bounded.

The purpose of this paper is to prove an analogue of Ingham's inequality for complex valued sequences with unbounded imaginary part, and to give some applications to null-controllability problems. Although the Ingham type inequalities proven here were motivated by problems in control theory, these results might also be of interest in operator theory and in general non-harmonic analysis.

In what follows, we assume

$$b_n \geq 0, \quad \forall n. \tag{1}$$

Theorem 1 *Fix $T > 0$. Suppose there exists a positive integer \tilde{p} and real number $r > 1$ such that for $k, n \geq \tilde{p}$,*

$$|a_n - a_k| \geq \gamma \max(1, b_n) |n^r - k^r|, \tag{2}$$

with γ some positive constant. Then there exist $C_1, C_2 > 0$ such that following inequalities hold for some positive integer $p \geq \tilde{p}$, and for all sequences $\{x_n\}$ with $\sum_{n=1}^\infty \frac{|x_n|^2}{1+b_n} < \infty$:

$$C_2 \sum_{n=p}^\infty \frac{|x_n|^2}{1+b_n} \geq \int_0^T |\sum_{n=p}^\infty x_n e^{i\omega_n t}|^2 dt \geq C_1 \sum_{n=p}^\infty \frac{|x_n|^2}{1+b_n}. \tag{3}$$

Note that for ω_n satisfying the hypotheses of this theorem, the family of exponentials $\{e^{i\omega_n t}\}$ will not be a Riesz basis of $L^2(0, T)$. Following Ingham, we will refer to the first inequality in Eq. 3 as the "direct inequality", and the second as the "inverse inequality".

We will show that sequences satisfying the hypotheses of Theorem 1 include $a_n = n^q$, with $q > 1$, and $b_n \leq \beta n^s$, with $s \in [0, q - 1)$ and β any positive constant. We also prove the following result:

Theorem 2 Fix $T > 0$. Suppose $|a_n - a_k| + b_n \rightarrow \infty$ as $\min(k, n) \rightarrow \infty$, and suppose there exists a positive integer \tilde{p} such that for $k, n \geq \tilde{p}$,

$$|a_n - a_k| \geq \gamma \max\left(\frac{\pi}{T}, b_n\right) |n - k|,$$

with

$$\gamma > \sqrt{10\pi}.$$

Then there exists $C > 0$ such that following inequality holds for some positive integer $p \geq \tilde{p}$ and for all sequences $\{x_n\}$ with $\sum_{n=1}^{\infty} \frac{|x_n|^2}{1+b_n^2} < \infty$:

$$\int_0^T \left| \sum_{n=p}^{\infty} x_n e^{i\omega_n t} \right|^2 dt \geq C \sum_{n=p}^{\infty} \frac{|x_n|^2}{1+b_n^2}. \quad (4)$$

Although the conclusions of this theorem are weaker than those for Theorem 1, this theorem does apply to sequences of the form ω_n with $a_n = n^q$, with $q > 1$, and $b_n \leq \frac{1}{\gamma} n^{q-1}$, whereas such sequences might not satisfy the hypotheses of Theorem 1.

Theorems 1 and 2 can be compared to a result that can be deduced from work of Hansen [7]. Hansen considers sequences $\{\omega_n\}$ for which there exists $\beta > 0$ such that

$$\frac{b_n}{|a_n|} \geq \beta, \quad (5)$$

together with the following separation conditions:

$$\begin{aligned} |\omega_n - \omega_k| &\geq \tilde{\delta} |n^r - k^r| && (r > 1, \tilde{\delta} > 0) \\ \epsilon(A + Bn^r) &\leq |\omega_n| < A + Bn^r. && (\epsilon > 0, A \geq 0, B \geq \tilde{\delta}). \end{aligned}$$

Hansen obtains from these hypotheses a sharp upper bound on the family of functions biorthogonal to $\{e^{i\omega_n t}\}$ on $L^2(0, \infty)$. From this bound can be deduced (see [15], Ch.2.Prop.4.1) the following inequality:

$$\int_0^T \left| \sum_{n=1}^{\infty} x_n e^{i\omega_n t} \right|^2 dt \geq \frac{C}{\sum_{n=1}^{\infty} 1/|\omega_n|} \sum_{n=1}^{\infty} \frac{|x_n|^2}{|\omega_n|} e^{-2b_n T}.$$

Notice that we have $b_n \rightarrow \infty$ here, so this inequality is much weaker than the inverse inequalities found in Theorems 1 and 2. However, it is easy to see that Eq. 2 is incompatible with Eq. 5, so the results in this paper are complementary to the one deduced from Hansen's work. It seems unlikely that the stronger inverse inequalities found in this paper can be proven for $\{\omega_n\}$ in the region given by Eq. 5. Indeed, in Example 3 of this paper, we adapt an example due to Ingham to show that for $\delta > 0$, the sequence $\omega_n = n + i\delta n$ does not satisfy the inverse inequalities in Eq. 3, 4, or even some weaker analogues.

The proofs of Theorems 1 and 2 are inspired by Ingham's original argument, in which the main object of analysis is the integral $\int_0^T g(t) \left| \sum_{n=1}^{\infty} x_n e^{i\omega_n t} \right|^2 dt$, where g is a carefully chosen continuous cutoff function which vanishes at $x = 0$ and $x = T$. In our proof of Theorem 2, we choose the truncated sine function originally used by Ingham, adapting his argument to complex frequencies. In our proof of Theorem 1, we set g equal to the characteristic function for $[0, T]$; thus in some sense we estimate $\int_0^T \left| \sum_{n=1}^{\infty} x_n e^{i\omega_n t} \right|^2 dt$ directly. These arguments differ from the methods used in

[1],[2], where essential use is made of the generating function associated with $\{\omega_n\}$. It appears that the latter methods, and those surveyed in [10], cannot easily be applied in our setting.

We then apply Theorems 1 and 2 to the null controllability of perturbations of the structurally damped beam equation with hinged ends. Let ω be an open subset of $(0, 1)$. Let $\rho > 0$, and ϖ, ϱ arbitrary reals. Consider the following system, where $\Delta = -\partial^2/\partial x^2$:

$$\begin{cases} u_{tt} + \Delta^2 u + \rho(\Delta^2)^\alpha u_t + \varpi u_{xx} + \varrho u = \chi_\omega f & \text{in } (0, 1) \times (0, T) \\ u(0, t) = g(t) & \text{in } (0, T), \\ u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } (0, 1). \end{cases}$$

Theorem 3 Fix $T > 0, \rho > 0$, and $\varpi, \varrho \in \mathbf{R}$. Assume $\alpha < 1/4$. Then

A) (Internal controllability). Assume $g = 0$. Suppose $u_0 \in L^2(0, 1)$, $u_1(x) \in (H^2(0, 1) \cap H_0^1(0, 1))'$. Then there exists a function $f = f_{(\rho, \varpi, \varrho)} \in L^2(\omega \times (0, T))$ such that $u(x, T) = u_t(x, T) = 0$. Also,

$$\int_0^T \int_\omega |f_{(\rho, \varpi, \varrho)}|^2 dx dt \leq C(\|u_0\|_{L^2(0, 1)}^2 + \|u_1\|_{(H^2(0, 1) \cap H_0^1(0, 1))'}^2),$$

with C independent of u_0, u_1 . Furthermore, for any $M > 0$, and for $\rho, \varpi, \varrho < M$, the constant C can be chosen independent of ρ, ϖ, ϱ .

B) (Boundary controllability). Assume $f = 0$. Suppose $u_0 \in H_0^1(0, 1)$, $u_1(x) \in H^{-1}(0, 1)$. Then there exists a function $g = g_{(\rho, \varpi, \varrho)} \in L^2((0, T))$ such that $u(x, T) = u_t(x, T) = 0$. Also,

$$\int_0^T |g_{(\rho, \varpi, \varrho)}|^2 dt \leq C(\|\frac{\partial}{\partial x}(u_0)\|_{L^2(0, 1)}^2 + \|(\frac{\partial}{\partial x}(-\Delta)^{-1}u_1)\|_{L^2(0, 1)}^2),$$

with C independent of u_0, u_1 . Furthermore, for any $M > 0$, and for $\rho, \varpi, \varrho < M$, the constant C can be chosen independent of ρ, ϖ, ϱ .

Furthermore, if $\alpha = 1/4$, then there exists $\rho_0 > 0$ such that $\rho < \rho_0$ implies the conclusions of Parts A and B.

The following follows from Theorem 3 by a standard argument:

Corollary 1 Fix $T > 0$. Assume $\alpha \in (0, 1/4]$.

A) (Internal Controllability) Fix initial conditions u_0, u_1 as in Theorem 1 part A. Let $f_{(\rho, \varpi, \varrho)} \in L^2(\omega \times (0, T))$ be the associated control function of minimal norm. Then, as $\rho + |\varpi| + |\varrho| \rightarrow 0^+$, $f_{(\rho, \varpi, \varrho)}$ converges in $L^2(\omega \times (0, T))$ to f_0 , where f_0 is a control function for the associated unperturbed beam equation.

B) (Boundary Controllability) Fix initial conditions u_0, u_1 as in Theorem 1 part B. Let $g_\rho \in L^2((0, T))$ be the associated control function of minimal norm. Then, as $\rho + |\varpi| + |\varrho| \rightarrow 0^+$, $g_{(\rho, \varpi, \varrho)}$ converges in $L^2((0, T))$ to g_0 , where g_0 is a control function for the associated unperturbed beam equation.

The system above with $f = g = 0$ and $\varpi = \varrho = 0$ is the one-dimensional version of the mathematical model for linear elastic systems with structural damping introduced by Chen and Russell in [5]. Internal null-controllability in time for arbitrarily small T for the undamped case ($\rho = 0$) was proven in [20]. Various results for boundary control in the undamped case can be found in [13], also see [2].

For a study of optimal boundary controllability of the damped plate equation, see [18]. Lasiecka and Triggiani [11] proved the null-controllability of the abstract equation:

$$w_{tt} + Aw + \rho A^\alpha w_t = u, \quad w(0) = w_0, w_t(0) = w_1, \rho > 0, \alpha \in [1/2, 1).$$

Here A is a strictly positive, self-adjoint unbounded operator with compact resolvent, and the control u is assumed to be distributed throughout $(0, 1)$. The proof for the case of $\alpha \in (0, 1/2)$ was published in followup papers [3],[16]. Although the framework of these papers is more general than that appearing in our Theorem 3A (at least for $\varpi = \varrho = 0$), their methods seem not to apply to the case where the control u is confined to a proper subset $\omega \subset (0, 1)$ as we do here. The paper [11] was motivated by questions in stochastic processes, and in [16] and [3], the authors address these questions by studying the rate of blowup of the optimal control as $T \rightarrow 0^+$. It would be interesting to make the corresponding study in our setting; however it seems not to be easy (see Remark 5 in this paper). In [7], Hansen studied the null-controllability of a damped vibrating rectangular plate equation, with $\alpha = 1/2$, subject to boundary control on one side, with $\rho < 2$, but his methods will also apply to the vibrating beam equation with internal damping to prove boundary null-controllability in time T for any $\rho < 2$. However, his proof, which uses a compactness argument, will not yield estimates on the control that are uniform in ρ .

Finally, the internal controllability of the structurally damped beam equation with hinged ends, and with $\varpi = \varrho = 0$, was studied by Edward and Tebou in [6] for the case $\alpha = 1/2$. In that setting, the conclusions of Theorem 3A and Corollary 1A were proven. The method of proof involved Ingham type inequalities similar to those in Theorem 2, but only applicable to a frequency band of finite width depending on ρ . The observability estimate for high frequencies used a generalisation of Bessel's inequality found in [9] (also see [12]), which in turn follows from a Carleman estimate. It seems likely that the methods used in [6] would also work to prove the conclusions of Theorem 3A for $\alpha \leq 1/2$ and arbitrary ϖ and ϱ , but they will not work for Theorem 3B because the generalised Bessel's inequality cannot be applied in the setting of boundary control. It should also be pointed out that the proof we have for Theorem 3A here is much simpler than the corresponding proof in [6].

This paper will be organized as follows. In Section 2, we will prove Theorems 1 and 2, and present some examples and a counterexample. Then in Section 3 we will prove Theorem 3.

2 Ingham Inequalities

Let K be a countable index set. Set

$$x(t) = \sum_{k \in K} x_k e^{i\omega_k t}. \tag{6}$$

To avoid questions of convergence, assume for the moment that only finitely many of the x_k 's are non-zero. Theorems 1 and 2 are then completed by a limiting argument, which we sketch at the end of their proofs. Let $g(t)$ be any real valued function and let $G(u) = \int_0^T e^{iut} g(t) dt$. Thus

$$\int_0^T g(t) |x(t)|^2 dt = \sum_{k \in K} \sum_{n \in K} x_k \overline{x_n} G(\omega_k - \overline{\omega_n})$$

$$\begin{aligned}
&= \sum_{n \in K} |x_n|^2 G(\omega_n - \bar{\omega}_n) + \sum_{n \in K} \sum_{k \in K, k \neq n} x_k \bar{x}_n G(\omega_k - \bar{\omega}_n) \\
&\equiv I + II.
\end{aligned} \tag{7}$$

In what follows we will study the conditions under which $(1 - \epsilon)|I| \geq |II|$ for some $\epsilon > 0$. Since

$$\omega_k - \bar{\omega}_n = (a_k - a_n) + i(b_k + b_n),$$

it is easy to see that $|G(\omega_k - \bar{\omega}_n)|$ is symmetric with respect to n and k . Thus,

$$\begin{aligned}
\left| \sum_n \sum_{k; n \neq k} x_k \bar{x}_n G(\omega_k - \bar{\omega}_n) \right| &\leq \frac{1}{2} \sum_n \sum_{k; n \neq k} (|x_k|^2 + |x_n|^2) |G(\omega_k - \bar{\omega}_n)| \\
&\leq \sum_n |x_n|^2 \sum_{k, k \neq n} |G(\omega_k - \bar{\omega}_n)|.
\end{aligned}$$

Comparing this last equation with Eq. 7, we see that if for some $\epsilon > 0$,

$$\sum_{k, k \neq n} |G(\omega_k - \bar{\omega}_n)| \leq (1 - \epsilon) |G(\omega_n - \bar{\omega}_n)|, \quad \forall n \in K, \tag{8}$$

then we have

$$\epsilon \sum_K |G(\omega_n - \bar{\omega}_n)| |x_n|^2 \leq \int_0^T g(t) |x(t)|^2 dt \leq (2 - \epsilon) \sum_K |G(\omega_n - \bar{\omega}_n)| |x_n|^2. \tag{9}$$

Proof of Theorem 1: We set $g(t) = 1$ on $[0, T]$. Hence, since $\omega_n - \bar{\omega}_n = 2ib_n$,

$$G(\omega_n - \bar{\omega}_n) = \begin{cases} \frac{1 - e^{-2b_n T}}{2b_n}, & b_n > 0 \\ T, & b_n = 0. \end{cases}$$

Now fix p for the moment, and for $n \geq p$ let

$$Q(n) = \sum_{k=p, k \neq n}^{\infty} \frac{1}{|k^r - n^r|}.$$

We will show that

$$\lim_{n \rightarrow \infty} Q(n) = 0. \tag{10}$$

Note first that since $r > 1$, $Q(n)$ absolutely convergent for each n . Also,

$$Q(n) = \sum_{j=1}^{\infty} \frac{1}{(n+j)^r - n^r} + \sum_{j=1}^{n-p} \frac{1}{n^r - (n-j)^r}.$$

The first series is easily seen to converge to zero as $n \rightarrow \infty$ by monotone convergence. For the second series we argue as follows. Simple arguments using the convexity of the function $x \rightarrow x^r$ imply that both for $j < 0$ and for $0 < j < n$,

$$\left| 1 - \left(1 - \frac{j}{n}\right)^r \right| \geq |j/n|. \tag{11}$$

Hence

$$\begin{aligned} \sum_{j=1}^{n-p} \frac{1}{n^r - (n-j)^r} &= n^{-r} \sum_{j=1}^{n-p} \frac{1}{1 - (1-j/n)^r} \\ &< n^{-r} \sum_{j=1}^{n-p} \frac{1}{j/n} \\ &< C n^{-r+1} \ln(n). \end{aligned}$$

Since $r > 1$, the vanishing of the second series follows.

Now choose p such that $n \geq p$ implies

$$Q(n) < \max\left(\frac{\gamma(1 - e^{-2T})}{8}, \frac{T\gamma}{4}\right). \quad (12)$$

Let $K = \{p, p+1, \dots\}$.

$$\begin{aligned} \sum_{k, k \neq n} |G(\omega_k - \bar{\omega}_n)| &\leq \sum_{k=p, k \neq n}^{\infty} \frac{|1 - e^{i(a_k - a_n)T - (b_n + b_k)T}|}{\sqrt{(a_n - a_k)^2 + (b_n + b_k)^2}} \\ &\leq \sum_{k=p, k \neq n}^{\infty} \frac{2}{\sqrt{(a_n - a_k)^2 + (b_n + b_k)^2}}. \end{aligned}$$

In the case $b_n > 1$, we have for $n \geq p$

$$\begin{aligned} \sum_{k=p, k \neq n}^{\infty} \frac{2}{\sqrt{(a_n - a_k)^2 + (b_n + b_k)^2}} &\leq \frac{1}{b_n} \sum_{k=p, k \neq n}^{\infty} \frac{2}{\gamma |n^r - k^r|} \\ &\leq \frac{1 - e^{-2b_n T}}{4b_n}. \end{aligned}$$

Now consider the case $b_n \leq 1$. A calculus exercise shows that $\frac{1 - e^{-2b_n T}}{2b_n}$ is a decreasing function of b_n on $[0, 1]$, and is thus has minimum value $(1 - e^{-2T})/2$. Hence for $b_n \neq 0$,

$$\begin{aligned} \sum_{k=p, k \neq n}^{\infty} \frac{2}{\sqrt{(a_n - a_k)^2 + (b_n + b_k)^2}} &\leq \sum_{k=p, k \neq n}^{\infty} \frac{2}{\gamma |n^r - k^r|} \\ &\leq \frac{1 - e^{-2T}}{4} \\ &\leq \frac{1 - e^{-2b_n T}}{4b_n}, \end{aligned}$$

while for $b_n = 0$,

$$\begin{aligned} \sum_{k=p, k \neq n}^{\infty} \frac{2}{\sqrt{(a_n - a_k)^2 + (b_n + b_k)^2}} &\leq \sum_{k=p, k \neq n}^{\infty} \frac{2}{\gamma |n^r - k^r|} \\ &< \frac{T}{2}. \end{aligned}$$

Thus in any case,

$$\sum_{k, k \neq n} |G(\omega_k - \bar{\omega}_n)| \leq \frac{1}{2} G(\omega_n - \bar{\omega}_n).$$

Noting that there exist $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$\frac{\tilde{C}_1}{1 + b_n} \leq G(\omega_n - \bar{\omega}_n) \leq \frac{\tilde{C}_2}{1 + b_n},$$

the theorem for K finite now follows easily from Eqs. 8, 9. In the case that K is infinite, we argue as follows. First, we note that $\sum_K x_n e^{i\omega_n t}$ is convergent in $L^2(0, T)$. This because the direct inequality for finite sequences implies

$$\int_0^T \left| \sum_{n=m_1}^{m_2} x_n e^{i\omega_n t} \right|^2 dt \leq C \sum_{n=m_1}^{m_2} \frac{|x_n|^2}{1 + b_n},$$

for positive constant C , and hence the sequence $m \rightarrow \sum_{n=p}^m x_n e^{i\omega_n t}$ is Cauchy in $L^2(0, T)$. Now set $K_m = \{n \in K; n \leq m\}$. Thus Eq. 3 holds for the set K_m . Letting $m \rightarrow \infty$, we obtain Eq. 3 for K . The proof of Theorem 1 is complete.

Remark 1: In the proof of Theorem 1, we could also choose $K = \{p, p+1, \dots\} \cup \{-p, -p-1, \dots\}$. In this case, the hypothesis should be that for $|k|, |n| \geq \tilde{p}$,

$$|a_n - a_k| \geq \gamma \max(1, b_n) |sgn(n)|n|^r - |sgn(k)|k|^r|.$$

Here

$$sgn(n) = \begin{cases} 1 & n > 0, \\ -1 & n < 0. \end{cases}$$

It is then easy to adapt the proof of Theorem 1 to this case. We will use this observation in Section 3.

Example 1 Set $a_n = n^q$ with $q > 1$ and $b_n \leq \beta n^s$ for some $\beta > 0$ and $s \in [0, q - 1)$. Let $r \in (1, q - s)$. We will show that there exists a constant $C > 0$ such

$$|n^q - k^q| \geq C n^s |n^r - k^r|; \quad (13)$$

Theorem 1 will then apply immediately. Trivially, Eq. 13 holds for $k = n$. Now let $C = q/(2r + s)$. For $n > k$, note that

$$\begin{aligned} (\partial/\partial_n - \partial/\partial_k)(n^q - k^q) &= q(n^{q-1} + k^{q-1}) \\ &= C(2r + s)(n^{q-1} + k^{q-1}) \\ &> C(r + s)n^{r+s-1} + Crn^s k^{r-1} \\ &> Csn^{s-1}(n^r - k^r) + Crn^s n^{r-1} + Crn^s k^{r-1} \\ &= (\partial/\partial_n - \partial/\partial_k)(Cn^s(n^r - k^r)). \end{aligned}$$

Eq. 13 now follows immediately for this case. In the case that $k > n$, we note that

$$\begin{aligned} (\partial/\partial_k - \partial/\partial_n)(k^q - n^q) &= q(k^{q-1} + n^{q-1}) \\ &\geq Cr(k^{r-1+s} + n^{r-1+s}) \\ &> Cr(n^s k^{r-1} + n^{r-1+s}) \\ &> -Csn^{s-1}(k^r - n^r) + Crn^s n^{r-1} + Crn^s k^{r-1} \\ &= (\partial/\partial_k - \partial/\partial_n)(Cn^s(k^r - n^r)). \end{aligned}$$

Remark 2: Let $K = \{1, 2, \dots\}$. If we choose $a_n = Ln^r$ with L sufficiently large and b_n as in Example 1, the arguments appearing there will yield Eq. 3 with $p = 1$.

Proof of Theorem 2: We apply the argument from the beginning of this section to a test function used by Ingham in [8]:

$$g(t) = \begin{cases} \sin(t\pi/T), & t \in [0, T], \\ 0, & \text{otherwise;} \end{cases} \quad (14)$$

thus its Fourier transform G satisfies

$$G(u) = \frac{\pi}{T} \frac{1 + e^{-iuT}}{\frac{\pi^2}{T^2} - u^2}.$$

Hence, in the notation of Eq. 7

$$I = \frac{\pi}{T} \sum_{n \in K} \frac{(1 + e^{-2b_n T})|x_n|^2}{\frac{\pi^2}{T^2} + 4b_n^2}. \quad (15)$$

We now estimate II. Choose $\epsilon > 0$ so $(1 - \epsilon)\gamma^2 > 5(2 + \epsilon)\pi$. Let $K = \{p, p + 1, \dots\}$, with p to be chosen below. Since $|a_n - a_k| + b_n \rightarrow \infty$ as $\min(k, n) \rightarrow \infty$, there exists P such that for $n \geq p$,

$$\sum_{k=p, k \neq n}^{\infty} \left| \frac{1 + e^{-i(a_n - a_k)T - (b_n + b_k)T}}{\frac{\pi^2}{T^2} - ((a_n - a_k) - i(b_n + b_k))^2} \right| \leq \sum_{k=p, k \neq n}^{\infty} \frac{2 + \epsilon}{(a_n - a_k)^2 + (b_n + b_k)^2}.$$

We now consider separately the cases $b_n \geq \pi/T$ and $b_n < \pi/T$. For $b_n \geq \pi/T$,

$$\begin{aligned} \sum_{k=p, k \neq n}^{\infty} \frac{2 + \epsilon}{(a_n - a_k)^2 + (b_n + b_k)^2} &\leq \frac{1}{b_n^2} \sum_{k=p, k \neq n}^{\infty} \frac{2 + \epsilon}{(\gamma|n - k|)^2 + (1 + b_k/b_n)^2} \\ &\leq \frac{4 + 2\epsilon}{b_n^2} \sum_{j=1}^{\infty} \frac{1}{\gamma^2 j^2} \\ &< \frac{1 - \epsilon}{5b_n^2} \\ &< \frac{1 - \epsilon}{\frac{\pi^2}{T^2} + 4b_n^2}. \end{aligned}$$

For $b_n < \pi/T$,

$$\begin{aligned} \sum_{k=p, k \neq n} \frac{2 + \epsilon}{(a_n - a_k)^2 + (b_n + b_k)^2} &\leq \frac{2 + \epsilon}{(\pi/T)^2} \sum_{k=p, k \neq n} \frac{1}{(\gamma|n - k|)^2} \\ &\leq \frac{4 + 2\epsilon}{(\pi/T)^2} \sum_{j=1}^{\infty} \frac{1}{\gamma^2 j^2} \\ &< \frac{1 - \epsilon}{5(\pi/T)^2} \\ &< \frac{1 - \epsilon}{\frac{\pi^2}{T^2} + 4b_n^2}. \end{aligned}$$

In both cases, we conclude that

$$\begin{aligned} II &\leq (1 - \epsilon) \frac{\pi}{T} \sum_{n \in K} \frac{|x_n|^2}{\frac{\pi^2}{T^2} + 4b_n^2} \\ &\leq (1 - \epsilon)I. \end{aligned}$$

The theorem for K finite now follows easily from Eqs. 8, 9. For K infinite, set $K_m = \{n \in K; n \leq m\}$. Thus Eq. 4 holds for the set K_m . Letting $m \rightarrow \infty$, we obtain Eq. 3 for K . The proof of Theorem 2 is complete.

Remark 3: With a weaker hypothesis on the constant γ , an equation of the form

$$\int_0^T \left| \sum_{n=p}^{\infty} x_n e^{i\omega_n t} \right|^2 dt \geq \epsilon \sum_{n=p}^{\infty} \frac{|x_n|^2}{1 + b_n^M}$$

with $M > 2$ can be proven. To prove such an inequality, the function k must be chosen to be more regular than the choice above (see [6]).

Example 2. Set $a_n = n^q$ with $q > 1$ and $b_n \leq n^{q-1}/\gamma$, with γ is in Theorem 2. Mimicking the arguments in Example 1 leading to Eq. 13, we get

$$|n^q - k^q| \geq n^{q-1}|n - k|.$$

If we choose p sufficiently large, then $n \geq p$ implies $n^{q-1} \geq \gamma\pi/T$. Since also $n^{q-1} \geq \gamma b_n$, it follows that

$$|n^q - k^q| \geq \gamma \max\left(\frac{\pi}{T}, b_n\right) |n - k|.$$

Theorem 2 now applies.

Example 3

The following counterexample is an adaptation of a counterexample due to Ingham [8]. Fix $T, \delta > 0$. We will show that for the sequence $\omega_n = n + i\delta n$, there exists no trio of positive constants m, p and C , with m, p integers, such that for the index set $K = \{p, p+1, \dots\}$,

$$\int_0^T \left| \sum_{n \in K} x_n e^{i\omega_n t} \right|^2 dt \geq C \sum_{n \in K} \frac{|x_n|^2}{n^{2m+1}}, \quad \forall \{x_n\} \in \ell^2(K). \quad (16)$$

Suppose such m, p and C exist. Let $H(z) = (1+z)^{-1/2}$, choosing the branch line as the ray $(-\infty, -1)$. Set

$$G(z) = H(z) - \sum_{n=0}^{m+p-1} \frac{H^{(n)}(0)z^n}{n!}.$$

Thus, G is holomorphic for $\Re z > -1$, and for $|z| < 1$ we have a Taylor expansion of the form

$$G(z) = \sum_{n=m+p}^{\infty} a_n z^n,$$

with $a_n = (2n)!/(2^n n!)^2$. Note that for $r < 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(re^{it})|^2 dt = \sum_{n=m+p}^{\infty} |a_n|^2 r^{2n}. \quad (17)$$

Let $g(z) = \frac{d^m}{dz^m}G(z)$. Thus g is holomorphic in $\Re z > -1$, and for $|z| < 1$ we have

$$g(z) = \sum_{n=p}^{\infty} b_n z^n,$$

with

$$b_n = a_{n+m} \cdot (n+m) \cdot (n+m-1) \cdot \dots \cdot (n+1).$$

Let

$$f(t) = g(re^{it-\delta t}) = \sum_{n=p}^{\infty} b_n r^n e^{it(n+i\delta n)}.$$

It is straightforward to calculate that

$$\begin{aligned} b_n &= \frac{a_{n+m}(n+m)!}{n!} \\ &= \frac{(2n+2m)!(n+m)!}{(2^{m+n}(n+m)!)^2 n!} \\ &\leq kn^m \end{aligned}$$

for some positive constant k , and hence $b_n r^n \in \ell^2(K)$ for each $r < 1$. Hence, if Eq. 16 were to hold, then we would have

$$\int_0^T |f(t)|^2 dt \geq C \sum_{n=p}^{\infty} \frac{|b_n|^2 r^{2n}}{n^{2m} + 1}. \quad (18)$$

Note that

$$\begin{aligned} \sum_{n=p}^{\infty} \frac{|b_n|^2 r^{2n}}{n^{2m} + 1} &\geq \sum_{n=p}^{\infty} \frac{|b_n|^2 r^{2n}}{((n+m) \cdot (n+m-1) \cdot \dots \cdot (n+1))^2} \\ &= \sum_{n=p}^{\infty} |a_{n+m}|^2 r^{2n} \\ &= r^{-2m} \sum_{n=m+p}^{\infty} |a_n|^2 r^{2n} \\ &= \frac{r^{-2m}}{2\pi} \int_{-\pi}^{\pi} |G(re^{it})|^2 dt. \end{aligned} \quad (19)$$

Combining Eqs. 18 and 19, we would get

$$\int_0^T |f(t)|^2 dt \geq \frac{Cr^{-2m}}{2\pi} \int_{-\pi}^{\pi} |G(re^{it})|^2 dt.$$

However, it is easy to see that $f(t)$ is continuous on $[0, T]$, uniformly in r for $r \in [0, 1]$. Thus the left hand side of the last equation is bounded as $r \rightarrow 1^-$, while it is easy to verify that the right hand side tends to infinity. This is absurd, hence C, p and m do not exist.

Remark 4: The author believes that Eq. 16 will also fail for $\omega_n = n + i\delta n^\alpha$ for any $\alpha > 0$.

3 Proof of Theorem 3

Proof of Theorem 3A: Assume for the moment that $\alpha < 1/4$. At the end of this section, we will indicate the simple modifications necessary for $\alpha = 1/4$. We use the well known HUM method of Lions [14],[13]. The associated adjoint equation is

$$\begin{cases} v_{tt} + \Delta^2 v - \rho(\Delta^2)^\alpha v_t + \varpi v_{xx} + \varrho v = 0 & \text{in } (0, 1) \times (0, T) \\ v(0, t) = v(1, t) = v_{xx}(0, t) = v_{xx}(1, t) = 0 & \text{in } (0, T) \\ v(x, T) = v_0(x), v_t(x, T) = v_1(x), & \text{in } (0, 1), \end{cases}$$

with $(v_0, v_1) \in [H^2((0, 1)) \cap H_0^1((0, 1))] \times L^2((0, 1))$. It will be convenient to apply Theorem 1 with parameter set $K = \{\pm p, \pm(p+1), \dots\}$ (see Remark 1). Thus we write the solution of the adjoint problem as

$$v(x, t) = \sum_{n \in \mathbf{Z} - \{0\}} \sin(n\pi x) c_n e^{\omega_n(t-T)},$$

with c_n determined by v_0, v_1 , and

$$\omega_n = \begin{cases} \frac{1}{2}(\rho n^{4\alpha} + \sqrt{\rho^2 n^{8\alpha} - 4(n^4 - \varpi n^2 + \varrho)}), & n > 0, \\ \frac{1}{2}(\rho |n|^{4\alpha} - \sqrt{\rho^2 |n|^{8\alpha} - 4(n^4 - \varpi n^2 + \varrho)}), & n < 0. \end{cases}$$

Note that since we assume $\rho, \varrho, \varphi < M$ and $\alpha < 1/4$, we can (and do) choose p such that $|n| \geq p$ implies $\rho^2 |n|^{8\alpha} - 4(n^4 - \varpi n^2 + \varrho) < 0$. Thus, in the notation of Remark 1, $a_n = \text{sgn}(n) \sqrt{n^4 - \varpi n^2 - \rho^2 |n|^{8\alpha}/4 + \varrho}$ and $b_n = \rho |n|^{4\alpha}/2$.

In what follows, we will denote by C various constants. Let $\|f\|^2 = \int_0^1 |f(x)|^2 dx$. Let

$$v_p(x, t) = \sum_{|n| \geq p} \sin(n\pi x) c_n e^{\omega_n(t-T)},$$

with p to be determined below. To prove Theorem 3A, it is well known that it suffices to prove the following observability estimate:

$$\int_\omega \int_0^T |(v_p)_t|^2 dt dx \geq C(\|\Delta v_p(*, 0)\|^2 + \|(v_p)_t(*, 0)\|^2), \quad (20)$$

with p and C independent of ρ . Since $\alpha < 1/4$, we have

$$\|\Delta v_p(*, 0)\|^2 + \|(v_p)_t(*, 0)\|^2 \leq C \sum_{|n| \geq p} n^4 |c_n|^2 e^{-\rho n^{4\alpha} T} \quad (21)$$

To bound $\int_\omega \int_0^T |(v_p)_t|^2 dt dx$ from below, first note that increasing p if necessary, $n \geq p$ implies there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$n^2(1 - \kappa_1/n^2) \leq a_n \leq n^2(1 + \kappa_2/n^2).$$

We conclude that, increasing p if necessary, $|n| \geq p$ implies there exists $\gamma > 0$ such that for $|n|, |k| \geq p$,

$$|a_n - a_k| \geq \gamma |\text{sgn}(n)n^2 - \text{sgn}(k)k^2|. \quad (22)$$

Applying the argument appearing in Example 1, we see that for $r \in (1, 2 - 4\alpha)$,

$$\begin{aligned} |a_n - a_k| &\geq Cn^{4\alpha} |\operatorname{sgn}(n)|n|^r - \operatorname{sgn}(k)|k|^r| \\ &\geq \max C(1, b_n) |\operatorname{sgn}(n)|n|^r - \operatorname{sgn}(k)|k|^r|. \end{aligned}$$

Hence, noting Remark 1, we apply Theorem 1 to conclude that with $K = \{p, p+1, \dots\} \cup \{-p, -(p+1), \dots\}$, we have

$$\begin{aligned} \int_{\omega} \int_0^T |(v_p)_t|^2 dt dx &= \int_{\omega} \left(\int_0^T \left| \sum_{n \in K} c_n \omega_n \sin(n\pi x) e^{\omega_n(t-T)} \right|^2 dt \right) dx \\ &\geq C \int_{\omega} \left(\sum_{n \in K} \frac{|\omega_n c_n \sin(n\pi x)|^2}{1 + \rho n^{4\alpha}/2} \right) dx \\ &\geq C \left(\sum_{n \in K} \frac{n^4 |c_n|^2}{1 + \rho n^{4\alpha}/2} \right). \end{aligned} \quad (23)$$

Finally, we observe that for each $T > 0$, there exists $C > 0$ such that

$$\frac{1}{1 + x/2} \geq C e^{-xT}, \quad \forall x > 0.$$

This, along with Eqs. 21 and 23, gives us Eq. 20.

Proof of Theorem 3B: the observability inequality associated with this system is:

$$\int_0^T \left| \frac{\partial v_p}{\partial x}(0, t) \right|^2 dt \geq C \left(\left\| \frac{\partial v_p}{\partial x}(*, 0) \right\|^2 + \left\| \frac{\partial}{\partial x} \left((-\Delta)^{-1} (v_p)_t(*, 0) \right) \right\|^2 \right), \quad (24)$$

with $(v_0, v_1) \in H_0^1((0, 1)) \times H^{-1}((0, 1))$. Note that the associated adjoint system is the same as that for Theorem 3A. We compute

$$\begin{aligned} \left\| \frac{\partial v_p}{\partial x}(*, 0) \right\|^2 + \left\| \frac{\partial}{\partial x} \left((-\Delta)^{-1} (v_p)_t(*, 0) \right) \right\|^2 &\leq C \sum_{|n| \geq p} (n^2 + |\omega_n|^2/n^2) |c_n|^2 e^{-\rho|n|^{4\alpha}T} \\ &\leq C \sum_{|n| \geq p} n^2 |c_n|^2 e^{-\rho|n|^{4\alpha}T}. \end{aligned} \quad (25)$$

Applying the same Ingham-type estimate as for the proof of Part A,

$$\begin{aligned} \int_0^T \left| \frac{\partial v_p}{\partial x}(0, t) \right|^2 dt &= \int_0^T \left| \sum_{|n| \geq p} n^2 \pi^2 c_n e^{\omega_n(t-T)} \right|^2 dt \\ &\geq C \sum_{|n| \geq p}^{\infty} \frac{n^4 |c_n|^2}{1 + \rho n^{4\alpha}/2}. \end{aligned} \quad (26)$$

The proof is completed by combining Eqs. 25 and 26.

Finally, we discuss the case $\alpha = 1/4$. In this case, we must adapt the argument in Example 2. Since $\beta_n = \rho|n|/2$, we have $\beta_n \leq n^{p-1}/\gamma$ provided $\rho \leq 2/\gamma$. Thus for $\rho \leq 2/\gamma$, the proofs of Theorem 3A and 3B can easily be mimicked, with Theorem 2 taking the place of Theorem 1.

Remark 5: As in [3],[16], it is natural to investigate the norm of the optimal control for 3A and 3B as $T \rightarrow 0^+$. To this end, we consider the dependence on T of various constants appearing in Theorem 1. The Ingham type estimate holds for the frequencies corresponding to $|n| \geq p$, but in view of Eq. 12, we would need $p = O(T^{-1})$ for the diagonal terms to dominate the off-diagonal terms in the proof of Theorem 1. In addition, it is easy to see that the constants C_1, C_2 appearing in Theorem 1 are both $O(1/T)$ as $T \rightarrow 0^+$. Thus for $|n| \geq p$, the norm of the optimal control is $O(1/T)$. Because $p \rightarrow \infty$ as $T \rightarrow 0^+$, it is unclear how to obtain a satisfactory bound on the norm for the frequencies corresponding to $1 \leq |n| \leq p$. Triggiani in [16], also see [17], obtains such an estimate for the case of a distributed control, but his argument does not seem to apply in our settings.

Acknowledgements. I wish to thank Louis Tebou for many helpful discussions.

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