Bounds on the number of resonances for the Laplacian on perturbations of certain spaces which are flat at infinity

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Abstract: The scattering theory for self-adjoint, elliptic perturbations of the Laplacian on $\mathbf{R}^n/\Gamma$, where $\Gamma \subset O(n)$ is a finite subgroup, is studied. Upper and lower bounds on the resonance counting function are proven. Certain asymptotically locally Euclidean spaces are also considered.

1 Introduction

Resonances play the role of eigenvalues for elliptic operators in non-compact domains and are defined as the poles of the meromorphically extended resolvent. They are connected with the local energy decay properties of the associated wave equation - see [21, 23] for surveys and references.

In this note we study the resonances associated to the Laplacian on compactly supported perturbations of the singular space,

$$\Omega_0 \equiv \mathbf{R}^n/\Gamma,$$

with Γ a finite subgroup of $O(n)$, $n \geq 2$. For odd dimensions, we also consider superexponentially decaying perturbations of metric.

Let ξ be the set of exceptional points, ie. points in $\mathbf{R}^n$ which are left fixed by some non-trivial subgroup of Γ. Let $\Pi : \mathbf{R}^n \rightarrow \Omega_0$ be the canonical projection. Let m be the cardinality of Γ. Thus, away from the set ξ, Π is an m-fold cover of Ω_0. We shall refer to $\Pi(\xi)$ as the singular set. We note in passing that the properties of the Laplacian on orbifolds, ie. quotients of the unit disk by Γ, have been studied in gauge theory.

One of the features of typical examples of these quotient spaces is that the singular set is unbounded. Thus, the results of this paper might be instructive in the study of other scattering problems with unbounded singular sets, such as the quantum N-body problem (see [16] and references therein). It should also be noted the similarities between the geometries of the N-body quantum problem and non-compact symmetric spaces [7].

This paper also extends the results in [1], where the scattering theory for perturbations of half-space with Dirichlet and Neumann boundary conditions is studied. There, the key free resolvent estimates are obtained using the Method of Images. The symmetry arguments used in this paper are somewhat different, but with the methods of this paper we can recover the results of [1] in the case of Neumann boundary conditions.

The results of this paper might also be of interest because they complement some of the results appearing in [20], where resonances of asymptotically locally Euclidean spaces are studied. A closer comparison between this paper and [20] will be made below.

We define

$$u \in C^{\infty}(\Omega_0) \Longleftrightarrow u \circ \Pi \in C^{\infty}(\mathbf{R}^n).$$

Let Δ_{R^n} be the Laplacian in $\mathbf{R}^n$:

$$\Delta_{R^n} = -\frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_n^2}.$$

We define the Laplacian on Ω_0 as follows: for $u \in C_0^{\infty}(\Omega_0)$,

$$\begin{aligned} \Delta_0 u(x) &= \Pi^* \Delta_{R^n} \Pi_* u(x) \\ &= \Delta_{\mathbf{R}^n}(u \circ \Pi)|_{\Pi^{-1}(x)}. \end{aligned} \tag{1}$$

Thus Δ_0 is well defined by the Γ-invariance of $\Delta_{\mathbf{R}^n}$. We define $L^2(\Omega_0)$ with the following norm:

$$\|u\|_0^2 = \frac{1}{m}\int_{\mathbf{R}^n} |u \circ \Pi|^2. \tag{2}$$

In Section 2 we show that Δ_0 extends from $C^{\infty}(\Omega_0)$ to a self-adjoint operator in $L^2(\Omega_0)$.

We now discuss the compactly supported perturbations of Δ_0. In what follows we employ the "black box" formalism used in [9]. The topological space on which the perturbed operator lives will be denoted by Ω, and we assume there exist compact sets $\kappa \subset \Omega$, $\kappa_0 \subset \Omega_0$, such that $\Omega - \kappa$ and $\Omega_0 - \kappa_0$ are identified as singular manifolds. Let $\mathcal{H}$ be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_\kappa \oplus L^2(\Omega - \kappa).$$

The corresponding orthogonal projections will be denoted by $u \to u|_{\Omega-\kappa}$ and $u \to u|_{\kappa}$. The operator $P : \mathcal{H} \to \mathcal{H}$ is assumed to be unbounded, self-adjoint with domain $\mathcal{D} \subset \mathcal{H}$. We also assume that

$$\mathcal{D}|_{\Omega-\kappa} \subset H^2(\Omega - \kappa),$$

and conversely, $u \in \mathcal{D}$ if $u \in H^2(\Omega - \kappa)$ and u vanishes near κ; also

$$(Pu)|_{\Omega-\kappa} = -\Delta_0(u|_{\Omega-\kappa}) \ \forall u \in \mathcal{D}; \tag{3}$$

$$I_\kappa (P+i)^{-1} \text{ is compact.} \tag{4}$$

Denote the set of bounded operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. We show:

Theorem 1 *Let $\chi \in C_0^\infty(\Omega)$. Then the mapping from $\{\Im(k) > 0\}$ to $\mathcal{L}(L^2(\Omega))$ given by*

$$k \to \chi(P-k^2)^{-1}\chi$$

extends to a meromorphic function in $\mathbf{C}$ for odd dimensions, and to a meromorphic function in Λ (the logarithmic plane) in even dimensions.

Theorem 1 is a simple adaptation of the analogous result in [9].

We define the resonances of P as the poles of $\chi(P-k^2)^{-1}\chi$ and we define the multiplicity of a resonance k_j as the dimension of the image of the projection

$$\frac{i}{\pi}\int_\gamma \chi(P-s^2)^{-1}\chi \ s ds,$$

with γ a sufficiently small loop around k_j. Let $\{k_j\}_{j\in\mathbf{N}}$ be the list of resonances including multiplicities.

We prove an upper bound on the number of resonances. Define

$$N(r) = \#\{k_j, \ |k_j| < r\}.$$

For even dimensions, it is more convenient to define

$$N(r,\alpha,\epsilon) = \#\{k_j, \ \epsilon < |k_j| < r, \ |\arg(k_j)| < \alpha\}.$$

The upper bounds on the resonances will be determined by the following two equations, in which we denote by $\mu_j(A)$ the characteristic values (see [2]) of the operator A. We assume that

$$\mu_j(\chi(P+i)^{-1}) \sim Cj^{-2/n}, \tag{5}$$

$$\mu_j([P,\tau](P+i)^{-1}) \sim Cj^{-1/n}. \tag{6}$$

Here χ is any smooth, compactly supported function, τ a C^∞ function whose gradient is compactly supported away from κ, and $[P,\tau]$ denotes the commutator of τ and P. C denotes constants depending on χ, τ. These estimates will hold, for instance, if Ω is a smooth manifold and $(P+i)^{-1}$ is a pseudodifferential operator of order -2 on Ω.

Theorem 2 *1- For $n \geq 3$ odd, there exists a constant $C > 0$ such that*

$$N(r) \leq Cr^n.$$

2- For n even and for any $\alpha \in (0, \pi)$, $\epsilon > 0$, there exists a constant $C > 0$ such that

$$N(r, \alpha, \epsilon) \leq Cr^n.$$

Theorem 2 is proven using the Fredholm determinant method (see [6, 17, 22]). The key estimates on the free resolvent, $(\Delta_0 - k^2)^{-1}$, come from observing that, because $\Delta_{\mathbf{R}^n}$ is invariant under rigid motions, it follows that the associated Green's function on Ω_0 is determined by the Green's function on $\mathbf{R}^n$.

The restrictions on α, ϵ in even dimensions are not necessary: the arguments in [18, 19] apply in our setting to give a bound on $N(r, \alpha, 0)$ which holds for all α. However, the proof is much easier for our, weaker, statement, which essentially follows from the proof of the odd dimensional case, after Jensen's Formula is replaced by a theorem of Carleman [5].

Under stronger geometric hypotheses, we can prove that the exponent in the upper bound of Theorem 2 is optimal. In what follows, we shall assume the existence of a periodic geodesic, although, if Ω is a manifold with boundary, the conclusions also follow from the existence of a periodic, non-glancing, geodesic that reflects off the boundary in a suitably defined way.

Theorem 3 *Assume that Ω satisfies the hypotheses of Theorem 1, and furthermore suppose there exists a single, elliptic, non-degenerate periodic geodesic in Ω. Then for any $N > 0$,*

$$\#\{k_j : 1 < |k_j| < r, |\Im z| < |z|^{-N}\} \geq C_N r^n.$$

The proof of this result will be sketched in Section 2. For exterior problems, analagous results have been proven in [12, 15, 11].

Since the key estimates in the results above depend on the invariance of $\Delta_{\mathbf{R}^n}$ under rigid motions, it is natural to ask whether there exist finite groups of rigid motions other than the finite subgroups of $O(n)$. In fact we shall prove in Lemma 9 of Section 3 that any finite subgroup of the group of rigid motions can be identified, after a translation of the origin, with a finite subgroup of $O(n)$. The proof of this result was kindly pointed out to the author by Steve Hudson.

We shall also consider non-compact perturbations of the metric on Ω_0, for n odd, thereby treating certain classes of asymptotically locally Euclidean spaces. To be more precise, suppose a metric $\tilde{g}_{ij}$ on $\mathbf{R}^n$ is Γ invariant. Associated to $\tilde{g}_{ij}$ is the Laplacian

$$\Delta_{\tilde{g}} = -\frac{1}{\sqrt{\tilde{g}}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \sqrt{\tilde{g}} (\tilde{g}^{i,j}) \frac{\partial}{\partial x_j}$$

and measure $\sqrt{\tilde{g}}$. Here $\tilde{g} = det(\tilde{g}_{ij})$ and $\tilde{g}^{ij} = (\tilde{g}_{ij})^{-1}$.

Now $\tilde{g}_{ij}$ naturally induces a Riemannian metric g_{ij} on $\mathbf{R}^n/\Gamma$ away from the singular set, namely $g_{ij} \equiv \Pi^* \tilde{g}_{ij}$. Defining $\Delta_g = \Pi^* \Delta_{\tilde{g}} \Pi_*$, it is clear that away from the singular part of $\mathbf{R}^n/\Gamma$, if we label the coordinates induced by Π by x_i,

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \sqrt{g} g^{i,j} \frac{\partial}{\partial x_j}.$$

We define $L^2_g(\Omega_0)$ with the norm

$$\|u\|_g^2 = \frac{1}{m} \int_{\mathbf{R}^n} |u \circ \Pi|^2 \sqrt{\tilde{g}} dx.$$

As with Δ_0, we can show that Δ_g is self-adjoint.

Assume now that the metric $\tilde{g}$ satisfies:

$$\begin{aligned} \tilde{g}_{ij} >> c_0 I,\ c_0 &> 0, \\ e^{A|x|^{1+\epsilon}} |\tilde{g}_{ij} - \delta_{ij}(x)| &\leq B, \end{aligned} \tag{7}$$

for fixed $c_0,\ \epsilon,\ A,\ B$.

Denote $< x >= (1+|x|^2)^{1/2}$. Denote the resolvents $(\Delta_g - k^2)^{-1}$, $(\Delta_{\tilde{g}} - k^2)^{-1}$ respectively by $R_g(k)$, $R_{\tilde{g}}(k)$ respectively. We shall show that

$$R_g(k) = \Pi^* R_{\tilde{g}}(k) \Pi_*. \tag{8}$$

The next two theorems follow immediately from the analogues proven by SaBarretto-Zworski in [8]:

Theorem 4 *The resolvent $R_g(k) : L^2(\Omega_0) \to H^2(\Omega_0)$, $\Im k > 0$, has a meromorphic continuation to $\mathbf{C}$:*

$$R_g(k) : e^{-<x>^{1+\delta}} L^2(\Omega_0) \to e^{<x>^{1+\delta}} H^2(\Omega_0)$$

for any $\delta > 0$. The poles are of finite rank and their multiplicity, given by the rank of the operator $\int_{\partial D(\lambda_0,\epsilon)} R_g(s) s ds$, is independant of δ.

Theorem 5 *For any $\delta > 0$, there exists a constant C_δ depending on the constants $\delta, \epsilon, A, B, c_0$, appearing in Eq. 7 and the Lipschitz constant of the g_{ij} such that*

$$N(r) \leq C_\delta r^{n(\max(\frac{1+\epsilon}{\epsilon}, \frac{3}{2} + \frac{1}{2\epsilon} + \delta))}.$$

Finally, in Section 2 we will prove:

Theorem 6 *Assume that Ω_0 satisfies the hypotheses of Theorem 5, and furthermore suppose there exists a single, elliptic, non-degenerate periodic geodesic in Ω. Then for any $N > 0$,*

$$\#\{k_j : 1 < |k_j| < r, |\Im z| < |z|^{-N}\} \geq C_N r^n.$$

We remark that Theorem 6 appears to be new even for $\Omega_0 = \mathbf{R}^n$, at least for those metrics not covered in [20].

The results above are somewhat complementary to Theorems 1,2,and 3 in [20]. In that work, Wunsch and Zworski use complex scaling techniques to study the resonances of the Laplacian on a class of spaces that include asymptotically locally Euclidean spaces. They prove the upper bound $N(r, 1, \alpha) \leq Cr^n$ for $\arg(\alpha)$ close to zero, and they also prove the conclusion of Theorem 6. They don't require their spaces to be asymptotic perturbations of quotient spaces, they require weaker estimates on the metric near infinity, and they don't require the dimension to be odd (which we do for Theorems 4, 5, and 6). On the other hand, their techniques require analyticity of the metric near infinity, and their meromorphic extension of resolvent (and hence their bounds on $N(r, 1, \alpha)$) holds only for $\arg(\alpha)$ close to zero.

We also remark that the methods in [22] suggest that the optimal upper bound for Theorem 5 should be $Cr^{n(1+\epsilon)/\epsilon}$.

This paper will be organised as follows. In Section 2 we prove the necessary resolvent estimates for Δ_0, along with Theorems 1, 2, and 6. Also we sketch the proofs of Theorem 3 and Eq. 8. In Section 3 we present various examples of model spaces.

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2 Proofs

We define the operator Δ_0 on $L^2(\Omega_0)$ as having operator domain

$$D(\Delta_0) = \{u \in L^2(\Omega_0) : \Delta_{\mathbf{R}^n}(u \circ \Pi) \in L^2(\Omega_0)\}.$$

Clearly Δ_0 is densely defined since $C_0^\infty(\Omega_0) \subset D(\Delta_0)$. In what follows, let $(*, *)_0$ denote the inner product in $L^2(\Omega_0)$, while $(*, *)_{\mathbf{R}^n}$ and $(*, *)_{D'}$ respectively denote the L^2 and distributional pairings in $\mathbf{R}^n$.

Lemma 1 *The operator Δ_0 is self-adjoint on $L^2(\Omega_0)$, with $C_0^\infty(\Omega_0)$ as a core.*

Proof: We show that $D(\Delta_0) = D(\Delta_0^*)$. It is a straightforward exercice to prove that Δ_0 is symmetric, so that $D(\Delta_0) \subset D(\Delta_0^*)$. To prove the converse, suppose $\phi \in D(\Delta_0^*)$, ie. there exists $\eta \in L^2(\Omega_0)$ such that

$$(\Delta_0 \psi, \phi)_0 = (\psi, \eta)_0, \ \forall \psi \in D(\Delta_0).$$

By Eqs. 2, 1, it follows that

$$\int_{\mathbf{R}^n} \Delta_{\mathbf{R}^n}(\overline{\psi \circ \Pi}) \ \phi \circ \Pi = \int_{\mathbf{R}^n} \overline{\psi \circ \Pi} \ \eta \circ \Pi.$$

Thus

$$(\psi \circ \Pi, \Delta_{\mathbf{R^n}}(\phi \circ \Pi))_{D'(\mathbf{R^n})} = (\psi \circ \Pi, \eta \circ \Pi)_{D'(\mathbf{R^n})}, \ \forall \psi \in D(\Delta_0). \tag{9}$$

Below, we shall show that the last equation holds with $\psi \circ \Pi$ replaced by arbitrary elements of $C_0^\infty(\mathbf{R}^n)$. It would then follow that $\Delta_{\mathbf{R^n}}(\phi \circ \Pi) = \eta \circ \Pi$ in the distributional sense, and hence in the L^2 sense. This would show that $\phi \in D(\Delta_0)$, completing the proof of the self-adjointness.

To show that Eq. 9 holds for arbitrary elements of $C_0^\infty(\mathbf{R^n})$, let $\varphi \in C_0^\infty(\mathbf{R^n})$. Since $\phi \circ \Pi$, $\eta \circ \Pi$, and $\sum_{g_i \in \Gamma} g_i^* \varphi$ are all Γ-invariant, it follows from Eq. 9 that

$$\begin{aligned}
(\varphi, \Delta_{\mathbf{R^n}}(\phi \circ \Pi))_{D'} &= \frac{1}{m}(\sum_{g_i \in \Gamma} g_i^* \varphi, \Delta_{\mathbf{R^n}}(\phi \circ \Pi))_{D'} \\
&= \frac{1}{m}(\sum_{g_i \in \Gamma} g_i^* \varphi, \eta \circ \Pi)_{D'} \\
&= (\varphi, \eta \circ \Pi)_{D'}.
\end{aligned}$$

To prove that $C_0^\infty(\Omega_0)$ is a core, let $\phi \in D(\Delta_0)$, so that $\phi \circ \Pi$ is a Γ-invariant element of $D(\Delta_{\mathbf{R}^n})$. Since $C_0^\infty(\mathbf{R}^n)$ is a core for $\Delta_{\mathbf{R}^n}$, it follows that there exists a sequence $u_n \in C_0^\infty(\mathbf{R}^n)$ such that we have L^2 convergence $u_n \to \phi \circ \Pi$ and $\Delta_{\mathbf{R}^n} u_n \to \Delta_{\mathbf{R}^n}(\phi \circ \Pi)$. Let

$$\tilde{u}_n = \frac{1}{m} \sum_{g_i \in \Gamma} g_i^* u_n.$$

Then we have L^2 convergence:

$$\tilde{u}_n \to \frac{1}{m} \sum_{g_i \in \Gamma} g_i^* (\phi \circ \Pi) = \phi \circ \Pi,$$

and by a similar argument we have $\Delta_{\mathbf{R}^n} \tilde{u}_n \to \Delta_{\mathbf{R}^n}(\phi \circ \Pi)$. Since $\tilde{u}_n$ are Γ invariant, we now have $L^2(\Omega_0)$ convergence: $\Pi^* \tilde{u}_n \to \phi$ and $\Delta_0 \Pi^* \tilde{u}_n \to \Delta_0 \phi$. Hence $C_0^\infty(\Omega_0)$ is a core.

We remark that if $u \circ \Pi$ is a C^∞ function, then Γ-invariance implies that $u \circ \Pi$ satisfies Neumann boundary condition on each of a finite number of subspaces (of varying dimension) whose union makes up the singular set ξ. Thus, in certain cases, Δ_0 can be expressed naturally as a Neumann Laplacian. Examples will be presented in Section 3.

Denote the resolvents $(\Delta_0 - k^2)^{-1}$ by $R_0(k)$, $(P - k^2)^{-1}$ by $R(k)$, $(\Delta_{\mathbf{R}^n} - k^2)^{-1}$ by $R_{\mathbf{R}^n}(k)$. Define the Sobolev spaces $H^i(\Omega)$ as the operator domains of $(P+1)^{i/2}$.

Lemma 2 $R_0(k) = \Pi^* R_{\mathbf{R}^n}(k) \Pi_*,$ *for* $\Im k < 0$.

Proof: The composition on the right hand side of the equation above is well defined because $R_{\mathbf{R}^n}(k)$ maps Γ-invariant functions to Γ-invariant functions. The lemma follows easily from Eq. 1.

Lemma 3 *Let n be odd (resp. even). Let $\psi_1, \psi_2 \in C_0^\infty(\Omega_0)$. Then the mapping from $\{\Im(k) > 0\}$ to $\mathcal{L}(L^2(\Omega_0))$ given by*

$$k \to \psi_2 R_0(k) \psi_1$$

extends to an entire function on $\mathbf{C}$ (resp. the logarithmic plane Λ). Also, the same is true for the mapping

$$k \to [\Delta_0, \psi] \psi_2 R_0(k) \psi_1,$$

for $i = 1, \ldots, n$, where $[\Delta_0, \psi]$ is the commutator of Δ_0 with any $\psi \in C_0^\infty(\Omega_0)$. Furthermore, for $\Im(k) \geq 0$ and $|k| > 1$,

$$\|\psi_2 R_0(k) \psi_1\|_{L^2(\Omega_0) \to H^i(\Omega_0)} \leq C_i (1 + |k|)^{i-1}, \; i = 0, 1, 2. \tag{10}$$

Finally, if $supp(\psi_1) \cap supp(\psi_2) = \emptyset$, then Eq. 10 holds for all i.

Proof: Observe first that $\|\Pi^*\|_{H^i(\Omega_0) \to H^i(\mathbf{R}^n)} = m$,$\|\Pi_*\|_{H^i(\mathbf{R}^n) \to H^i(\Omega_0)} = 1/m$. It follows from Lemma 2 that

$$\|\psi_1 R_0(k) \psi_2\|_{L^2(\Omega_0) \to H^i(\Omega_0)} \leq \|\psi_1 \circ \Pi \; R_{\mathbf{R}^n}(k) \; \psi_2 \circ \Pi\|_{L^2(\mathbf{R}^n) \to H^i(\mathbf{R}^n)}.$$

Since $\psi_j \circ \Pi$ are compactly supported, the estimates for $\|\psi_1 R_0(k) \psi_2\|$ follow from those for $\mathbf{R}^n$ proven in [22] for n odd, and [3], [4],[17],[18] for n even. The rest of the lemma is proven similarly. $\Box$

The proof of Theorem 1 now follows, word for word, from the proof the analagous result in [1], which itself is a simple adaptation of an argument found in [9]. For the reader's convenience we sketch it here. In what follows, let $\kappa_i \subset \Omega$ be open sets with compact closure, for $i = 1, 2, 3, 4$, with $\kappa \subset \kappa_1 \subset \kappa_2 \subset \kappa_3 \subset \kappa_4$. We define a smooth partition of unity $\chi_1 + \chi_2 = 1$ such that $\chi_i \geq 0$, $supp(\chi_1) \subset \kappa_2$, and $\chi_1 = 1$ on κ_1. We also define smooth cutoff functions $\tau_i \geq 0$ such that $\tau_1 = 1$ on $supp(\chi_1)$ and $supp(\tau_1) \subset \kappa_3$, and $\tau_2 = 1$ on $supp(\chi_2)$ and $\tau_2 = 0$ on κ. Finally, we define a cutoff function $\rho \in C_0^\infty(\mathbf{R}^n)$ such that the $supp(\rho) \subset \kappa_4$ and

$$\rho|_{\kappa_3} = 1. \tag{11}$$

We define an approximation of $R(k)$ as follows. Let

$$R_a(k) = \tau_1 R(k_0) \chi_1 + \tau_2 R_0(k) \chi_2.$$

Here k_0 is a parameter to be chosen below.

We have

$$(P-k^2)R_a(k) = I + K,$$

with

$$K = (k_0^2 - k^2)\tau_1 R(k_0)\chi_1 + [P,\tau_1]R(k_0)\chi_1 + [P,\tau_2]R_0(k)\chi_2. \tag{12}$$

By Eq. 2 we have for $\Im k > 0$:

$$R_a = (P-k^2)^{-1}(I+K).$$

By Eqs. 12, 11 we have $\rho K = K$, hence

$$R_a\rho = (P-k^2)^{-1}\rho(I+K\rho).$$

For $k = k_0$ and $Im(k_0) >> 0$, we have by the Spectral Theorem that $\|K\rho\|_{L^2\to L^2} < 1$ and hence we can write

$$\rho R_a(k)\rho(I+K\rho)^{-1} = \rho R(k)\rho.$$

Fix such a k_0. Now $\rho R_a(k)\rho$ is an entire function of k with values in $\mathcal{L}(L^2(\Omega))$, and so meromorphy of $\rho R(k)\rho$ is equivalent to meromorphy of $(I+K\rho)^{-1}$. The meromorphy of the latter is then proven using the Meromorphic Fredholm Theorem.

Proof of Theorem 2 :

In what follows, let C be various positive constants. A simple Neumann series argument shows that $(I+K\rho)$ is invertible if and only if $(I-(-K\rho)^{n+1})$ is invertible. To apply the Fredholm determinant method, first we must prove that $(K\rho)^{n+1}$ is trace class. The standard proof of this (eg. [1]) does not apply because K is not necessarily a pseudodifferential operator.

Lemma 4 *The operator $(K\rho)^{n+1}$ is trace class.*

Proof: We begin by writing

$$K\rho = K_1 + K_2$$

with $K_2 = [P,\tau_2]R_0(k)\chi_2\rho$. We will show that $K_2 \in \mathcal{S}_{n+1}$, where $\mathcal{S}_j$ is the Schatten class of order j (see [2]). By Eqs. 5, 6 the same is true for the first two terms in Eq. 12. The lemma will then follow immediately.

Denote by $L^2_\Gamma(\mathbf{R}^n)$ the set of Γ invariant functions in $L^2(\mathbf{R}^n)$. Note that $\frac{1}{m}\Pi_* : L^2(\Omega_0)\to L^2_\Gamma(\mathbf{R}^n)$ is a unitary transformation with inverse $m\Pi^*$. On the other hand, $[P,\tau_2] = [\Delta_0,\tau_2]$, so

$$\begin{aligned}\Pi_* K_2 &= [\Pi^*\circ\Delta_0\circ\Pi_*, \tau_2\circ\Pi,]R_{\mathbf{R}^n}(k)(\chi_2\rho)\circ\Pi \\ &= [\Delta_{\mathbf{R}^n}, \tau_2\circ\Pi]R_{\mathbf{R}^n}(k)(\chi_2\rho)\circ\Pi.\end{aligned}$$

Since $[\Delta_{\mathbf{R}^n}, \tau_2 \circ \Pi]$ is a differential operator of order 1 on $\mathbf{R}^n$, $\Pi_* K_2$ is pseudodifferential operator of order -1 on $\mathbf{R}^n$, with compactly supported Schwartz kernel. It follows that $\Pi_* K_2$ is in $\mathcal{S}_{n+1}$ when viewed as an operator from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$. It then follows that $\Pi_* K_2$ is in $\mathcal{S}_{n+1}$ when viewed as an operator from $L^2_\Gamma(\mathbf{R}^n)$ to $L^2_\Gamma(\mathbf{R}^n)$. By unitary invariance, $K_2 : L^2(\Omega_0) \to L^2(\Omega_0)$ is in $\mathcal{S}_{n+1}$. $\Box$

It follows that the Fredholm determinant $det(I - (-K\rho)^{n+1})$ is entire on $\mathbf{C}$ for n odd, and on Λ for n even, and furthermore:

Lemma 5 *The resonances of Δ (counted with their multiplicities) are among the zeros of the function*

$$k \to h(k) \equiv det(I - (-K\rho)^{n+1}(k)),$$

counted with their multiplicities.

The reader is referred to the appendix in [19] for a proof of this result.

The rest of the proof of Theorem 2 proceeds much as in [1]. We sketch the argument here, indicating the necessary modifications in our setting. We assume for the moment that n is odd, after which the modifications necessary for n even will be indicated.

Because of Lemma 5, the bound appearing in Theorem 2 will follow from Jensen's inequality and the estimate

$$|h(k)| \leq C \exp(C|k|^n). \tag{13}$$

To obtain this estimate, we apply the theory of characteristic values developed in [2], and adapted to exterior problems in [6, 22, 17]. Denote the characteristic values of a compact operator A, listed in increasing order and including multiplicities, by $\{\mu_j(A)\}$.

Let K_1, K_2 be as in the proof of Lemma 4. Applying inequalities on Fredholm determinants appearing in [2], we get

$$det(I-(-K\rho)^{n+1}) \leq (\prod_{j=1}^{\infty}(1+2^n\mu_j(|K_1|)^{n+1}))^{2n+2}(\prod_{j=1}^{\infty}(1+2^n\mu_j(|K_2|)^{n+1}))^{2n+2}. \tag{14}$$

Using Eqs. 5, 6 and arguing exactly as in [1], we get

$$\prod_{j=1}^{\infty}(1 + 2^n\mu_j(|K_1|)^{n+1}) \leq e^{C|k|^n}, \tag{15}$$

this estimate holding for all $k \in \mathbf{C}$.

Next, we estimate the terms involving K_2. The argument sketched below is a simple adaptation of one originally implemented by Vodev in [17].

Lemma 6 *Suppose $\Im k \geq 0$. Then $det(I + 2^n|K_2|^{n+1}) < C$ for some positive constant C.*

Proof: We have, using standard Weyl asymptotics along with Eq. 10 with $i = 3$,

$$\begin{aligned} \mu_j(K_2) &= \mu_j(\rho(I+P)^{-1}(I+P)K_2) \\ &\leq \mu_j(\rho(I+P)^{-1})\|(I+\Delta_0)K_2)\| \\ &\leq Cj^{-2/n}|k|^2. \end{aligned}$$

Now the arguments leading to Eq. 15 are easily adapted to this case. The details are omitted.

Lemma 7 *Suppose $\Im k < 0$. Then*

$$det(I + 2^n|K_2|^{n+1}) \leq \exp(C|k|^n).$$

Proof: We use the notation of Lemma 3. Arguing as in Lemma 3 of [1], it suffices to estimate

$$det(I + 2^n|K_2(k) - K_2(-k)|^{n+1})^{2n+2}.$$

To estimate this term, we note, as in [1] that

$$\rho(R_{\mathbf{R}^n}(k) - R_{\mathbf{R}^n}(-k))\rho = c_n k^{n-2}\rho E^*(k)E(k)\rho, \tag{16}$$

with $E(k)$ being a mapping from $L^2(\mathbf{R}^n)$ to $L^2(S^{n-1})$ whose Schwartz kernel is given by

$$e^{-ik<\omega,y>}, \text{ for } \omega \in S^n, y \in \mathbf{R}^n. \tag{17}$$

Denote $\tilde{\rho} = \rho \circ \Pi$. Hence

$$\rho(R_0(k) - R_0(-k))\rho = c_n k^{n-2}\Pi^*\tilde{\rho}E^*(k)E(k)\tilde{\rho}\Pi_*.$$

Observe that the Schwartz kernel is an entire function in k, and we have the estimate

$$\|\Pi^*\tilde{\rho}E^*(k)\|_{L^2(S^n)\to L^2(\Omega_0)} \leq Ce^{C|\Im k|},$$

with C some positive constant. Arguing as in [1], we also have that

$$\mu_j(E(k)\tilde{\rho}\Pi_*) \leq e^{-j^{1/(n-1)}/C}e^{C|k|}.$$

Hence

$$\mu_j(\rho(R_0(k) - R_0(-k))\rho) \leq e^{-j^{1/(n-1)}/C}e^{C|k|}.$$

A similar argument yields, for $\Im k < 0$

$$\mu_j\left([\Delta, \tau_2](R_0(k) - R_0(-k))\chi_2\rho\right) \leq e^{-j^{1/(n-1)}/C}e^{C|k|}.$$

The proof of the lemma is now completed exactly as in [1]. $\square$

Combining Eqs. 14, 15 and Lemmas 6 and 7, we obtain Eq. 13 and hence Theorem 2 for n odd.

We now complete the proof for n even. In place of Jensen's formula, we use Carleman's Theorem ([5], Thm.5.1.1). The bounds on the determinant for large k are proven as in the odd dimensional case (set the branch line on the negative real axis in Lemma 7), and the bounds for k near the origin are obtained simply by analyticity.

Proof of Equation 8: First, we have

Lemma 8 *The operator Δ_g with operator domain*

$$D(\Delta_g) = \{u \in L^2_g(\Omega_0) : \Delta_{\tilde{g}}(u \circ \Pi) \in L^2_g(\Omega_0)\}$$

is self-adjoint on $L^2_g(\Omega_0)$.

Proof: The proof of this is the same, word for word, as the proof of Lemma 1.

Eq. 8 follows immediately from Lemma 8.

Proof of Theorem 6:

This theorem is proven by first proving an *apriori* estimate for the resolvent near the real axis and away from the resonances. Such a bound was first proven for exterior problems by Zworski in unpublished work. The *apriori* estimate then enables us to apply the quasimode-to-resonance argument found in [11]. The use of quasimode arguments together with an *apriori* estimate on the resolvent has been used, for compactly supported perturbations of operators on $\mathbf{R}^n$, to prove the existence of resonances [13, 12], and then lower bounds on the number of resonances [15, 11].

We shall in fact prove the result for metric perturbations of $\mathbf{R}^n$; the corresponding result for quotient spaces follows immediately from Eq. 8.

In what follows, we denote $R_{\mathbf{R}^n}, R_{\tilde{g}}$ and $L^2(\mathbf{R}^n)$ by R_0, R, and L^2 respectively. For $N \in \mathbf{R}$ we define

$$\chi_N(x) = \chi(|x|/(S|N|^{1/\epsilon})), \; < x >_N \equiv \chi_N(x)(1+|x|^2)^{1/2},$$

where S is a large, fixed positive constant, $\chi \in C_0^\infty(\mathbf{R})$, $0 \le \chi \le 1$, $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$, and ϵ as in Eq. 7. We define weighted L^2-spaces by $L^2_N(\mathbf{R}^n) \equiv e^{N<x>_N} L^2(\mathbf{R}^n)$. In what follows, we will denote these spaces simply as L^2_N, and we will be assuming $N >> 0$.

We now prove the *apriori* upper bound for $R(k)$ in a conic neighbourhood of the real axis and away from the resonances.

In what follows, we will denote by C various positive constants that are independent of N, k. Denote by $Y(r)$ a positive, differentiable function on $\mathbf{R}^+$

that vanishes as $r\to\infty$ and with $Y'(r) \in (-1,0)$. Denote the resonances of $\Delta_{\tilde{g}}$ by $\{k_j\}$, and let

$$\Lambda(r,M,\alpha) = \{k : M < |k| < r, |\arg k| < \alpha\}.$$

Denote by $B(k,r)$ a disk centred at k of radius r, and denote $B(k_j, Y(|k_j|))$ by B_j. We will show that there exist $\alpha > 0$, $n^{\#}$, such that for $k \in \Lambda(\infty, M, \alpha) - \cup B_j$,

$$\|R(k)\|_{L^2(\mathbf{R}^n, e^{<x>^{1+\epsilon}}dx)\to L^2(\mathbf{R}^n, e^{<x>^{-1-\epsilon}}dx)} \le e^{C|k|^{Cn^{\#}}\log(\frac{|k|^2}{Y(|k|)})} \tag{18}$$

Now, by making the substitutions $k = \sqrt{z}/h$ and $k_j = \sqrt{z_j}/h$, a simple calculation shows that for any $\tilde{\Omega} \subset\subset \Lambda(\infty, M, \alpha)$, the inclusion $z \in \tilde{\Omega} - \cup B(z_j, DhY(D/h))$ implies

$$\|(h^2P - z)^{-1}\|_{L^2(\mathbf{R}^n, e^{<x>^{1+\epsilon}}dx)\to L^2(\mathbf{R}^n, e^{<x>^{-1-\epsilon}}dx)} \le e^{\tilde{D}h^{-\tilde{D}n^{\#}}\log(\frac{\tilde{D}}{h^2Y(1/h)})};$$

here $D, \tilde{D}$ are positive constants that depend on $\tilde{\Omega}$ but not on N, k. Setting $g(h) = DhY(D/h)$, we arrive at the conclusion of Lemma 1 in [11]. The arguments of that paper now apply in our setting, and Theorem 6 of this paper is an immediate conclusion.

To prove Eq. 18, we begin with a representation of the resolvent in this setting:

$$R(k) = (R_0(k) - R(k_0)QR_0(k))(I + (k^2 - k_0^2)R(k_0)QR_0(k))^{-1};$$

here $Q = \Delta_{\tilde{g}} - \Delta_{\mathbf{R}^n}$. This representation is used in [8] to prove that for γ any small, fixed positive number and N is any positive integer, $R(k)$ extends to a map $L^2_{-N}\to L^2_N$ for $\Im k > -N + \gamma$. We shall prove that there exist $C_1 \in (0,1)$, $\alpha \in (0, \pi/2)$, and $M > 0$ such that for $k \in \Lambda(C_1N, M, \alpha) - \cup B_j$,

$$\|(R_0(k) - R(k_0)QR_0(k))\|_{L^2_{-N}\to L^2_N} \le e^{C|k|^2}, \tag{19}$$

$$\|(I + (k^2 - k_0^2)R(k_0)QR_0(k))^{-1}\|_{L^2_{-N}\to L^2_{-N}} \le e^{C|k|^{Cn}\log(C|k|^2/Y(|k|))} \tag{20}$$

Combining Eqs. 19, 20 we get

$$\|R(k)\|_{L^2_{-N}\to L^2_N} \le e^{C|k|^{Cn}\log(C|k|^2/Y(|k|))}$$

for $k \in \Lambda(C_1n, M, \alpha) - \cup B_j$. Since we have that the inclusions:

$$\begin{aligned} L^2(\mathbf{R}^n, e^{<x>^{1+\epsilon}}dx) &\to L^2_{-N}, \\ L^2_N &\to L^2(\mathbf{R}^n, e^{<x>^{-1-\epsilon}}dx) \end{aligned}$$

both have norm less than or equal to $e^{CN^{1+1/\epsilon}}$, it follows that

$$\|R(k)\|_{L^2(\mathbf{R}^n,e^{<x>^{1+\epsilon}}dx)\to L^2(\mathbf{R}^n,e^{<x>^{-1-\epsilon}}dx)} \leq e^{|k|^{Cn}\log(C|k|^2/Y(|k|))+CN^{1+1/\epsilon}}$$

for $k \in \Lambda(C_1N, M, \alpha) - \cup B_j$. Setting $n^{\#} = \max(1+1/\epsilon, Cn)$ and increasing M if necessary, Eq.18 now follows.

It remains to prove Eqs. 19, 20. Eq. 19 follows from the decomposition

$$R_0(k) = R_0(-k) + (R_0(k) - R_0(-k)).$$

In fact, by Lemma 1 in [8], we have for $\Im k < 0$

$$\|R_0(-k)\|_{L^2_{-N}\to L^2_N} \leq C\frac{N^{1/\epsilon}}{1+|k|}$$

for $|k| < \tilde{C}_1 N$, with $\tilde{C}_1 \in (0,1)$ independant of N. Hence, there exists $M_1 > 0$ such that $M_1 < |k| < \tilde{C}_1 N$ implies

$$\|R_0(-k)\|_{L^2_{-N}\to L^2_N} \leq C\frac{|k|^{1/\epsilon}}{1+|k|}$$

To bound the term $R_0(k) - R_0(-k)$, note first that

$$\|R_0(k) - R_0(-k)\|^2_{L^2_{-N}\to L^2_N} \leq \int_{x,y} |S(x,y)|^2 e^{-2N<x>_N - 2N<y>_N} dx\ dy,$$

where $S(x,y)$ is the Schwartz kernel of $R_0(k) - R_0(-k)$. Now by Eqs. 16, 17, which still hold when the terms ρ are removed, we have

$$S(x,y) = k^{n-2}\int_{S^{n-1}} e^{ik<w,x-y>}dw.$$

A straightforward calculation thus shows that for $|k| < \tilde{C}_1 N$,

$$\begin{aligned}\|R_0(k) - R_0(-k)\|^2_{L^2_{-N}\to L^2_N} &\leq C|k|^{2n-4}\left(\int_{|x|<CN} e^{2|\Im k||x|}dx\right)^2 \\ &\leq e^{CN^2}.\end{aligned}$$

For large k, the term N appearing on the right hand side of of the last inequality can be replaced by $|k|$. Eq. 19 now follows.

Finally we prove Eq. 20. Set $\tilde{B} = (k^2-k_0^2)R(k_0)QR_0(k)$, and $B = e^{N<x>_N}\tilde{B}e^{-N<x>_N}$. Applying an elementary argument followed by the theory of Fredholm determinants [2],

$$\begin{aligned}\|(I+\tilde{B})^{-1}\|_{L^2_{-N}\to L^2_{-N}} &= \|(I+B)^{-1}\|_{L^2\to L^2} \\ &\leq |\det_{L^2}(I+B^n)|^{-1}|\det_{L^2}(I+|B|^n)|^n. \qquad (21)\end{aligned}$$

In the proof of Proposition 2 in [8], it is shown that

$$|\det_{L^2}(I+B^n)|^n \leq e^{CN^{Cn}}$$

for $|k| \leq \tilde{C}_2 N$, with $\tilde{C}_2 \in (0,1)$. Inspection of the proof shows that the same estimate will hold with B replaced by $|B|$. It follows that there exists $M_3 > 0$ such that for $M_3 < |k| \leq \tilde{C}_2 N$,

$$f(k) \equiv |\det_{L^2}(I+B^n)|^n \leq e^{C|k|^{Cn}}, \tag{22}$$

with the same estimate also holding with B replaced by $|B|$. Recall now the mini-modulus theorem of Cartan (see [5], p.79), which states that if g is analytic in $B(0,2eR)$ and $|g(0)| > 0$, then one has

$$|g(z)| > |g(0)| \left(\max_{|z|=2eR} |g(z)/g(0)| \right)^{-H(\eta)}, \quad H(\eta) = \log(15e^3/\eta),$$

this estimate valid in $B(0,R)$ outside a system of disks whose summed radii is less that ηR.

Choose $k_0 \in \mathbf{R}^+$ such that $|f(k_0)| > 0$. Applying the Cartan lemma iteratively with $\eta < 1$ fixed, one then obtains the following lower bound for f for any $l > k_0$:

$$|f(k)| > |f(k_0)| e^{-CH(\eta)|k|^{Cn}};$$

this estimate is valid in $B(l, l - M_3/2e)$ outside a system of disks whose summed radii is less than $l - M_3/2e$, and with C is some universal constant. In what follows, set $\tilde{l} = l - M_3$. Now we apply the Cartan lemma again, with $\eta = l^{-2}Y(l)$. Thus in $B(l, \tilde{l}/2e)$ but outside of a system of disks with summed radii less than $Y(l)/l$, we have

$$|f(k)| > |f(k_0)| e^{-C|k|^{Cn} \log(C|k|^2/Y(|k|)}.$$

Thus in $B(l, \tilde{l}/2e)$ but outside of the system of disks, we have

$$|\det_{L^2}(I+B^n)|^{-1} < e^{C|k|^{Cn} \log(C|k|^2/Y(|k|)}. \tag{23}$$

with C a universal constant.

We decompose the system of disks into the union $\cup U_j$, where U_j are connected, mutually disjoint, and have diameter at most $l^{-1}Y(l)$. By shrinking the disk $B(l, \tilde{l}/2e)$ slightly if necessary, we also have that $U_j \subset B(l, \tilde{l}/2e)$. Then we can assume that each U_j contains a resonance. For if not, then Eq. 23 holds on U_j by the maximum principle. Assume without loss of generality that $B(l, \tilde{l}/2e) \subset \{|k| > 1\}$. Thus for each j, $U_j \subset B(k_i, Y(|k_i|))$ for some resonance $k_i \in B(l, \tilde{l}/2e)$. Thus, there exists $\alpha \in (0, 1/2e)$ such that Eq. 23 holds for $k \in \Lambda(\tilde{C}_3 N, 1, \alpha) - \cup B_j$, for some $\tilde{C}_3 \in (0, \tilde{C}_2)$. Thus Eq. 20 now follows by combining Eqs. 21, 22, and 23, and setting $C_1 = \min(\tilde{C}_1, \tilde{C}_3)$, and $M = \max(M_1, M_2, M_3, 1)$.

Proof of Theorem 3: This is proven by following the arguments used in the proof of Corollary 1.3 in [1], except that the lower bound on $f(k)$ is obtained as in the proof of Theorem 6 of this paper, rather than as in [1] where the mini-modulus theorem of Cartan is improperly applied. As in the proof of Theorem 6 above, we arrive at the conclusion of Lemma 1 in [11]. Then arguing as in [11], the conclusions of Theorem 3 of this paper follow. The details are omitted.

3 Examples

1- Half spaces, quarter spaces, etc. Let Γ be the two element subgroup of $O(2)$ generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus $\mathbf{R}^2/\Gamma$ is the half space $\mathbf{H}^2$. It is easily checked that Δ_0 is the Laplacian on half-space with Neumann boundary conditions, and the same is true for Δ_g. For the Euclidean metric, this has previously been treated in [1]. For metrics g satisfying Eq. 7, the conclusions of Theorems 4-6 complement those of [20] (see discussion in the introduction of this paper).

Similarly, one can choose Γ so that the associated Δ_0 is the Neumann Laplacian for quarter spaces, eighth spaces, etc, for arbitrary dimension.

2- Two dimensional cone. Let Γ be the m-element group generated by

$$\begin{pmatrix} \cos(2\pi/m) & \sin(2\pi/m) \\ -\sin(2\pi/m) & \cos(2\pi/m) \end{pmatrix}.$$

Thus $\mathbf{R}^2/\Gamma$ is a cone with vertex angle $\tan^{-1}(1/m)$, and Δ_0, away from the vertex, is the Laplace Beltrami operator associated to the metric induced by Π. Let $r = \sqrt{x^2 + y^2}$, and denote $\Pi^* r$ again by r. Then an operator core is the compactly supported, C^2 functions f such the $\partial f/\partial r \to 0$ as $r \to 0$.

3- Let Γ be the m-element group generated by

$$\begin{pmatrix} \cos(2\pi/m) & \sin(2\pi/m) & 0 \\ -\sin(2\pi/m) & \cos(2\pi/m) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $\mathbf{R}^3/\Gamma$ is a space that is identical, as a singular manifold, to $\aleph \times \mathbf{R}$, where $\aleph$ is the space generated in Example 2.

4- Let Γ be the subgroup of $Gl(3)$ generated by $-I$. Then $\mathbf{R}^3/\Gamma$ is a cone with cross-section $\mathbf{RP}^2$, and Δ_0 has, as in Example 2, an operator core of C_0^∞ functions satisfying Neumann boundary conditions at the vertex.

We conclude this section by proving that the methods of this paper cannot be extended to more general groups of isometries on $\mathbf{R}^n$. Note first that we need the isometry group to be finite for the pullback Π_* to be L^2-bounded. Next, we have the following lemma, whose proof was pointed out to the author by Steve Hudson:

Lemma 9 *Suppose Γ is a finite group of rigid motions on $\mathbf{R}^n$. Then there exists $p \in \mathbf{R}^n$ and a finite group $\tilde{\Gamma} \subset O(n)$ such that*

$$\Gamma = \{T_p A T_{-p};\ A \in \tilde{\Gamma}\};$$

here T_p is the translation $T_p(x) = x + p$.

Proof: Let m equal the cardinality of Γ. For $x \in \mathbf{R}^n$, let $x_1, x_2, \ldots, x_m$ be the orbit of Γ through x. Set $p = \sum_1^m x_i/m$. We show that p is a fixed point for Γ. In fact, for $g \in \Gamma$, it is clear that the list $\{g(x_1), \ldots, g(x_m)\}$ is a permutation of $\{x_1, \ldots, x_m\}$. Thus, decomposing the rigid motion as $g = g_0 T_q$, with $q \in \mathbf{R}^n$ and $g_0 \in O(n)$,

$$\begin{aligned} g(p) &= g_0 \sum_1^m (x_i + q)/m \\ &= \sum_1^m g(x_i)/m \\ &= \sum_1^m x_i/m \\ &= p. \end{aligned}$$

It follows that the rigid motion $T_{-p} g T_p$ has the origin as a fixed point, and hence $T_{-p} g T_p \in O(n)$. Since the mapping $g \rightarrow T_{-p} g T_p$ is a group isomorphism from Γ into a subgroup of $O(n)$, the lemma now follows.

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