Lecture 10 Rayleigh and Mie Scattering

Review of some scattering/absorption (extinction) parameters (see Lec 6&7 for details):

1. Scattering/absorption coefficient:
   \[ \beta_{s\lambda}, \beta_{a\lambda} \quad \text{Unit: } m^{-1} \]

2. Mass scattering/absorption coefficient (Specific scattering/absorption cross-section)
   \[ k_{s\lambda}, k_{a\lambda} \quad \text{Unit: } \frac{\text{area}}{\text{mass}} \]
   \[ \beta_{s\lambda} = \rho k_{s\lambda}, \quad \beta_{a\lambda} = \rho k_{a\lambda} \]
   Where \( \rho \) is the density of the medium.

3. Extinction coefficient:
   \[ \beta_{e\lambda} = \beta_{s\lambda} + \beta_{a\lambda} \]
   \( \beta_{e\lambda} \) is the rate of power attenuated per unit distance due to extinction.

4. Mass extinction coefficient
   \[ k_{e\lambda} = k_{a\lambda} + k_{s\lambda} \quad \text{and} \quad \beta_{e\lambda} = \rho k_{e\lambda} \]
   \( k_{e\lambda} \) has dimensions of area per mass. \( k_{e\lambda} \) is also considered as an extinction cross-section (\( \sigma_{e\lambda} \), see definition below) per unit mass.

Some new scattering/absorption (extinction) parameters:

1. Scattering/absorption cross section: \( \sigma_{a\lambda}, \sigma_{s\lambda} \)
   Previously, the mass scattering/absorption coefficient was defined by finding the proportionality between the volume scattering/absorption coefficient (\( \beta_{a\lambda}, \beta_{s\lambda} \)) and the mass density \( \rho \) of the medium. Now the scattering/absorption cross section (\( \sigma_{a\lambda}, \sigma_{s\lambda} \)) is defined by relating the volume scattering/absorption coefficient (\( \beta_{a\lambda}, \beta_{s\lambda} \)) to the number density (concentration \( N_p \)) of particles within the medium:
   \[ \beta_{a\lambda} = N_p \sigma_{a\lambda}, \quad \beta_{s\lambda} = N_p \sigma_{s\lambda} \]
   The constants of above proportionalties \( \sigma_{a\lambda}, \sigma_{s\lambda} \) has dimensions of area (m\(^2\)). \( \sigma_{a\lambda}, \sigma_{s\lambda} \) are referenced to a single particle, not a mass of medium as \( \beta_{s\lambda}, \beta_{a\lambda} \) or \( k_{s\lambda}, k_{a\lambda} \).

2. Extinction cross-section:
   \[ \sigma_{e\lambda} = \sigma_{s\lambda} + \sigma_{a\lambda} \]
   The extinction cross-section \( \sigma_{e\lambda} \) can be thought of as the equivalent area of an opaque object blocking the same total amount of radiation. \( \sigma_{e\lambda} \) is different with the extinction coefficient \( k_{e\lambda} \) in that \( \sigma_{e\lambda} \) is referenced to a single particle while \( k_{e\lambda} \) is referenced to a unit mass. Since \( k_{e\lambda} \) is an extinction cross per unit mass, we have:
   \[ \sigma_{e\lambda} = k_{e\lambda} m \]
   where \( m \) is the mass per particle

3. Scattering and absorption efficiencies:
   \[ Q_{s\lambda} \equiv \frac{\sigma_{s\lambda}}{A}, \quad Q_{a\lambda} \equiv \frac{\sigma_{a\lambda}}{A} \]
   Where \( A \) is the geometric cross-sectional area of the particle. In case of a spherical droplet, \( A = \pi r^2 \), where \( r \) is the radius.
4. Extinction efficiency:  
\[ Q_{e\lambda} = Q_{s\lambda} + Q_{a\lambda} \]

In the visible and IR band, a single cloud droplet has an extinction cross-section that is similar to, but not identical to its geometric cross-section \( A \). It is possible that a particle can extinguish more radiation than its geometric cross-section would imply. In another word, \( Q_{e\lambda} \) could be greater than 1!

Relative index of refraction:

\[ m = \frac{N_2}{N_1} \]

Where \( N_1 \): complex refractive index of the particle; \( N_2 \): complex refractive index of the surrounding medium (which is the air for meteorology). Recall \( N = n + i n_i \). Note that \( n \) governs the phase speed of propagation of a wave within the material; \( n_i \) governs absorption.

\( N_1 \) is usually taken to be equal to 1 for air, so \( m \) approximates to \( N_2 \).

\( N_2 \) depends on both the composition of the particles and on the wavelength \( \lambda \).

Rayleigh Scattering:

Size parameter \( (x = \frac{2\pi r}{\lambda}) \) range: \( x < 0.2 \) (or sometimes referred to as \( x \ll 1 \))

Rayleigh Solution: to answer how much radiation a small particle scatters and/or absorbs in the atmosphere. The solution for Rayleigh extinction efficiency \( Q_e \) is (a complex number):

\[ Q_e = 4\pi l_m \left\{ \frac{m^2 - 1}{m^2 + 2} \left[ 1 + \frac{x^2}{15} \left( \frac{m^2 - 1}{m^2 + 2} \right) \left( \frac{m^4 + 27m^2 + 38}{2m^2 + 3} \right) \right] \right\} + \frac{8}{3} x^4 R \left( \frac{m^2 - 1}{m^2 + 2} \right)^2 \]

\[ Q_s = \frac{8}{3} x^4 \left| \frac{m^2 - 1}{m^2 + 2} \right|^2 \]

Where \( l_m \{ \} \) means taking the imaginary part, and \( R\{ \} \) means taking the real part, \( x \) is size parameter, \( m \) is the relative index of refraction of the scattering particles.

Since \( Q_a = Q_e - Q_s \), for sufficiently small \( x \),

\[ Q_a = 4\pi l_m \left\{ \frac{m^2 - 1}{m^2 + 2} \right\} \]

Interpretation:

1) \( Q_a \propto x; Q_s \propto x^4 \), the absorption efficiency is proportional to size parameter \( x \), while the scattering efficiency is proportional to \( x^4 \).

2) For sufficiently small \( x \), assuming \( m \) has non-zero imaginary part: \( Q_s \ll Q_a \) & \( Q_a \approx Q_e \)

3) The single scattering albedo \( \omega = \frac{Q_s}{Q_e} \propto x^3 \), is proportional to \( x^3 \).
Key facts about scattering and absorption in Rayleigh Regime:

1) \( \sigma_s = Q_sA = \pi r^2 \cdot Q_s \), \( \sigma_s \propto r^2 \cdot x^4 \sim \frac{r^6}{\lambda^4} \). scattering cross section is proportional to \( \frac{r^6}{\lambda^4} \).

Therefore: A): if you have radiation of fixed wavelength \( \lambda \) and use it to illuminate two particles of radius \( r_1 < r_2 \), the larger particle will scatter the radiation more strongly by a factor of \( \left( \frac{r_2}{r_1} \right)^6 \).

   —— This is of central importance of weather radar (larger particles produce much larger reflectivity than smaller particle).

B): If you have a particle of fixed size and expose it to radiation with two different wavelengths \( \lambda_1 < \lambda_2 \), then it will scatter the shorter wavelength \( \lambda_1 \) more strongly by a factor of \( \left( \frac{\lambda_2}{\lambda_1} \right)^4 \).

   —— This fact explains why the sky is blue (blue light has smaller wavelength than other colored lights in visible band).

2) For sufficiently small particles with complex (not pure real) refractive index \( m \), scattering is negligible and absorption is proportional to mass path only, independent on particle size.

   —— This fact is relevant to passive microwave remote sensing of cloud water, where we don’t need to consider particle size variations.

   since \( k_a = \frac{\text{area}}{\text{mass}} = \frac{\sigma_a}{\text{mass}} = \frac{Q_a \pi r^2}{\rho \cdot \left( \frac{4}{3} \right) \pi r^3} \)

   \[ = \frac{3Q_a}{4\pi r} \cdot \frac{6\pi}{\rho \lambda} \cdot \frac{m^2 - 1}{m^2 + 2} \]

   \[ \Rightarrow k_a \text{ is independent on } r. \]

Where \( r \) is the radius of the water droplet, \( \rho \) is the density of water

**Mie Solution:** (for a spheric particle)

\[ Q_e = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n + 1)R(a_n + b_n) \]

\[ Q_s = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n + 1)(|a_n|^2 + |b_n|^2) \]

Where the coefficients \( a_n \) and \( b_n \) are Mie Scattering Coefficients and are functions of \( x \) and \( m \). \( n \) is any integer.
Fig. 12.4 from Petty’s textbook: The extinction efficiency $Q_e$ as a function of size parameter $x$ for a non-absorbing sphere with relative refraction index $m=1.33$ for various ranges of $x$. (a) “Big picture” view showing that $Q_e \to 2$ as $x \to \infty$. (c) Detail for $x<0.8$, comparing the Rayleigh (small particle) approximation and exact Mie theory.

Explanation Petty’s textbook Fig. 12.4:

a) —— Extinction efficiency $Q_e$ increases as size parameter $x$ increases up to $x = 6$, where $Q_e = 4$; after $x > 6$, $Q_e \to 2$ as $x \to \infty$ (extinction efficiency $Q_e$ oscillates around 2 as size parameter $x$ increases up to infinity)

b) ——Rayleigh Approximation is good up to $x = 0.6$ for a non-absorbing sphere with relative refraction index $m=1.33$. 
