An Introduction to Limits

MAC 2311

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The concept of a "limit" is the fundamental building block on which all calculus concepts are based. This section is intended to present an informal view of limits with the ultimate goal of developing an intuitive understanding for the basic ideas.

1 An Intuitive Approach to Limits

1.1 Rates of Change

One of the most important characteristics of functions is how a function behaves or *changes* over time. For example, we may be interested in how much the value of a function changes over a certain interval. This leads to a quantity known as the **average rate of change** of the function.

Given any function, $y = f(x)$, we find the **average rate of change** of y with respect to x over an interval from x_1 to x_2 by taking the ratio of the change in y,

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$$

and the change in x,

$$\Delta x = x_2 - x_1.$$

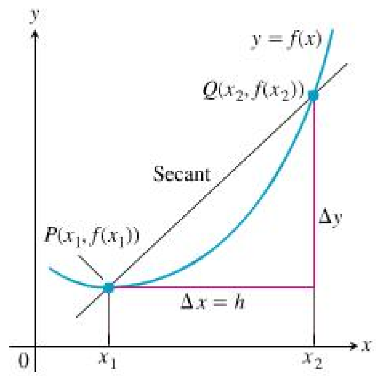
So, the average rate of change of a function $y = f(x)$ over an interval $[x_1, x_2]$ is found by:

$$\textbf{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

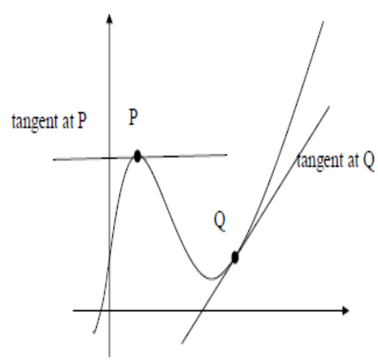
Example 1: Find the average rate of change of $f(x) = \sin(x)$ over the interval $\left[0, \frac{\pi}{6}\right]$.

Example 2: A rock is launched vertically upward from the ground with a speed of 85 ft/sec. Not considering air resistance, the position of the rock after t seconds is given by $s(t) = -16t^2 + 85t$. find the average rate of change of the rock between 1 and 3 seconds.

Let's say we have a function $f(x)$ that contains two points, P: (x_1, y_1) and Q: (x_2, y_2), on its curve. Geometrically, the slope of the line between two points on a curve gives us the average rate of change between the two points. A line through two points on a curve is called a **secant line**. The slope of a secant line is the **average rate of change** between the two points.



A line passing through a point on a curve that has the same direction as the curve there is called the **tangent line to the curve at that point**. What, then, is the direction of the curve at a point? Given the geometry of a tangent line to a curve at a single point, it makes sense to define **the direction of the curve at a point** as the slope we find when we let the secant line points get as close together as possible if that slope exists. This slope, the limit of the secant line slopes as one point gets arbitrarily close to the point we chose for the tangent line, is the instantaneous rate of change at a single point.



So, the average rate of change gives the rate of change of a function across an interval while the instantaneous rate of change gives the rate of change of a function at one particular point.

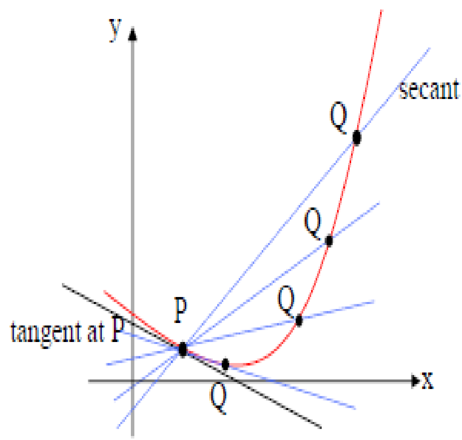
1.2 Connecting Average and Instantaneous Rates of Change

Finding a Tangent Line

To find the tangent line at a point P:

- Imagine there are two points on a curve, P and Q.
- Let the point Q move closer to the point P along the curve.
- As Q moves closer to P, the secant lines between P and Q begin to approach the tangent line at P.
- So, as Q gets closer to P, the slopes of the secant lines between P and Q approaches the slope of the tangent line at P (As $Q \to P$, $m_{PQ} \to m_P$)

The slope of the tangent line is the **limit** of the slopes of the secant lines as one point approaches the other.



Finding Instantaneous Rate of Change

Let's again consider how we find the average rate of change between two points on a curve. If we compute the average rate of change of $y = f(x)$ over the interval $[x_0, x_0 + h]$, then the change in x is represented by h. In other words, $\Delta x = (x_0 + h) - x_0 = h$. So, the average rate of change is calculated as $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + h) - f(x_0)}{h}$. To find the instantaneous rate of change at x_0, the change in x would have to be zero in the average rate of change. However, we see that we cannot substitute in $h = 0$ since we cannot divide by zero.

So, if we want to find the instantaneous rate of change of a function $f(x)$ at a point $x = x_0$,

- Compute the average rates of change over intervals that get shorter and shorter (approaching the point of interest, x_0).
- As the length of intervals decrease, the average rates of change begin to approach a unique number.
- This unique number is the instantaneous rate of change at x_0.

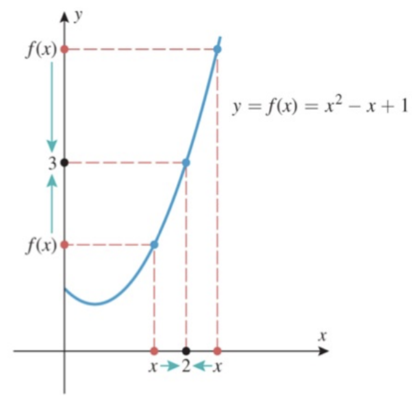
The instantaneous rate of change is the value the average rate of change approaches as h, the change between the two points, approaches zero.

Example 3: Determine the instantaneous rate of change of $f(x) = x^2$ at $x = 3$.

- Find the average rates of change over the given time intervals. Do not round your intermediate calculations.
  - $[3, 4]$
  - $[3, 3.5]$
  - $[3, 3.1]$
  - $[3, 3.01]$
  - $[3, 3.001]$
- Estimate the instantaneous rate of change when $x = 3$.

2 The Limit of a Function

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let's examine the behavior of the function $f(x) = x^2 - x + 1$ as x-values get closer and closer to 2. We can see from both the graph and the table of function values, as x approaches 2 from the left and right sides, the function values approach 3.



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

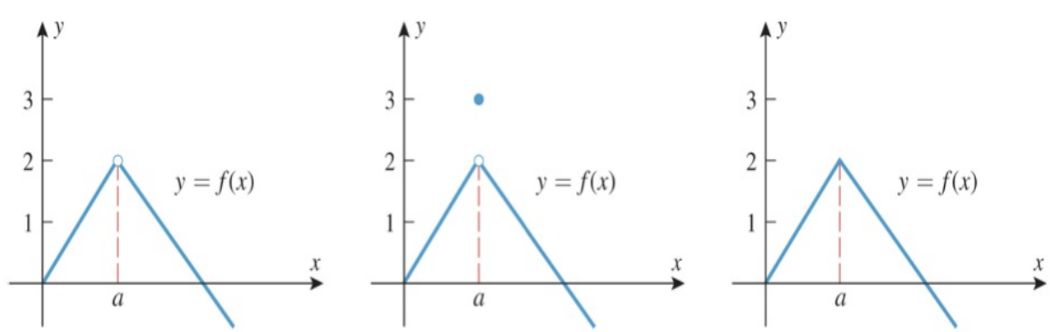
Left side → ← Right side

We would describe this by saying "the limit of $x^2 - x + 1$ as x approaches 2 is 3".

<u>Limit Notation</u>: For a function $f(x)$, as x gets arbitrarily close to a, if $f(x)$ gets arbitrarily close to L, then we say that "the limit of $f(x)$ as x approaches a is L". We use the following notation to express this:

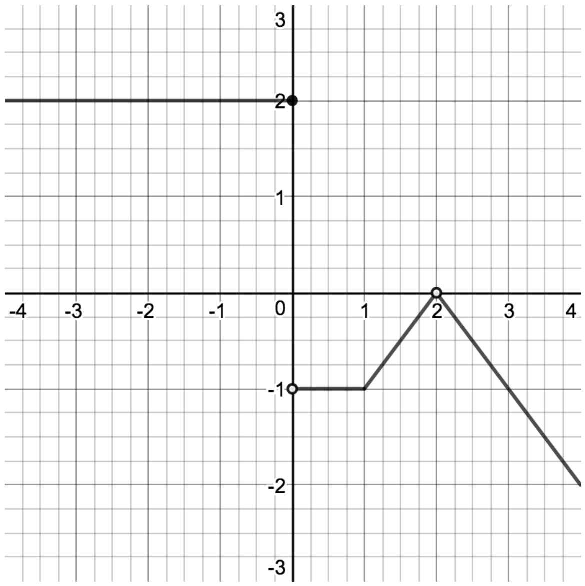
$$\lim_{x \to a} f(x) = L$$

It is important to note that limits only tell us how a function is behaving *near* a point and not what is happening *at* the point. Take the following for example:



We notice that in each of the graphs illustrated, the limit of $f(x)$ as x approaches a is 2, or $\lim_{x \to a} f(x) = 2$. However, in the far left graph $f(a)$ is undefined and in the middle graph $f(a) = 3$. Again, the limit of $f(x)$ as x gets arbitrarily closer to a only tells us how the function is behaving near a, not what is actually happening at $x = a$.

Example 4: Evaluate the following limits using the graph of $f(x)$ given below.



- $\lim_{x \to -1} f(x) =$
- $\lim_{x \to 1} f(x) =$
- $\lim_{x \to 2} f(x) =$

2.1 One-Sided and Two-Sided Limits

The limit, $\lim_{x \to a} f(x) = L$, is referred to as a **two-sided limit** because it describes what happens as a function approaches an x value from both sides, left and right. However, some functions exhibit behaviors that differ on either side of an x-value. In this case, it is necessary to indicate whether x-values near a point are on the left side or right side of the point in order to determine limiting behavior.

<u>One-Sided Limits</u>

For a function $f(x)$, as x gets arbitrarily close to a, but always *less than* a, if $f(x)$ gets arbitrarily close to L_1, then we say "the limit of $f(x)$ as x approaches a from the left is L_1". We use the following notation to express this **left-hand limit**:

$$\lim_{x \to a^-} f(x) = L_1$$

For a function $f(x)$, as x gets arbitrarily close to a, but always *greater than* a, if $f(x)$ gets arbitrarily close to L_2, then we say "the limit of $f(x)$ as x approaches a from the right is L_2". We use the following notation to express this **right-hand limit**:

$$\lim_{x \to a^+} f(x) = L_2$$

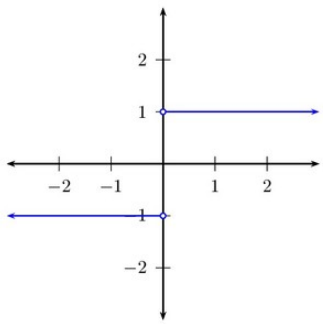
Note: The superscript of "$-$" indicates a limit from the left of $x = a$ and a superscript of "$+$" indicates a limit from the right of $x = a$.

Relating One-Sided and Two-Sided Limits

The two-sided limit of a function $f(x)$ exists at $x = a$ if and only if both of the one-sided limits exist at a and have the same value. Therefore, if the right- and left-hand limits do not agree (are not the same value), the two-sided limit fails to exist. That is,

$$\lim_{x\to a} f(x) = L \textbf{ IF AND ONLY IF } \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

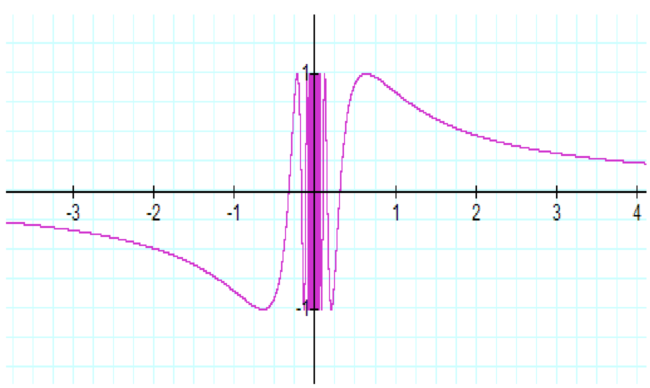
For example, consider the function $f(x) = \dfrac{|x|}{x}$. From the graph of $f(x)$ (see below), we are able to see that $\displaystyle\lim_{x\to 0^-} \frac{|x|}{x} = -1$, while $\displaystyle\lim_{x\to 0^+} \frac{|x|}{x} = 1$. Since the left- and right-hand limits do not agree, the two-sided limit at $x = 0$ will fail to exist for $f(x) = \dfrac{|x|}{x}$.



In general, when the limit of a function fails to exist at a point, we use the expression of "DNE" to indicate that it "does not exist". So, for the above example, we would say: $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$ DNE.

When Limits Fail to Exist

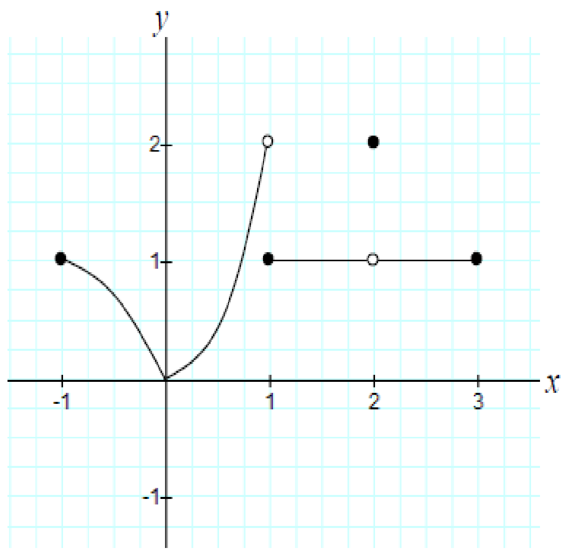
- If $\displaystyle\lim_{x\to a^-} f(x) = L_1$ and $\displaystyle\lim_{x\to a^+} f(x) = L_2$, where $L_1 \neq L_2$, then $\displaystyle\lim_{x\to a} f(x)$ fails to exist.
  - If $\displaystyle\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\displaystyle\lim_{x\to a} f(x)$ DNE.
- If the function oscillates too much around a point, then the limit of the function at that point will not exist.
  - For example, consider the function $f(x) = \sin\left(\dfrac{1}{x}\right)$



x	0.01	0.001	0.0001	0.00001	0.000001
$\sin\left(\frac{1}{x}\right)$	-0.5064	0.8269	-0.3056	0.0357	-0.3450

From the graph of $f(x)$, it's difficult to tell what is happening near $x = 0$. So, let's look at a table of function values of $f(x)$ at x-values getting closer to 0. We see that as $x \to 0^+$, the function values "jump around" never approaching one unique number. Because of this, the limit as x approaches 0 fails to exist. Or, $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ DNE.

Example 5: Use the graph of $f(x)$ below to determine the following values, if they exist.



- $f(1)$
- $\lim\limits_{x\to 1^-} f(x)$
- $\lim\limits_{x\to 1^+} f(x)$
- $\lim\limits_{x\to 1} f(x)$
- $f(2)$
- $\lim\limits_{x\to 2^-} f(x)$
- $\lim\limits_{x\to 2^+} f(x)$
- $\lim\limits_{x\to 2} f(x)$