Indeterminate Forms; L'Hôpital's Rule

MAC 2311

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When evaluating limits, we've found that direct substitution can lead to various invalid expressions. Some of the expressions have the form of a number, but values cannot be assigned to them by using the basic rules of division, multiplication, or powers. Since we are not able to determine an actual value from expressions of this form, we are led to refer to them as **indeterminate forms**.

Indeterminate Forms

The following expressions are indeterminate forms:

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad \pm\infty\cdot 0 \qquad \infty-\infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

L'Hôpital's Rule

If you are evaluating the limit of a rational expression as x approaches a (a can be finite or infinite):

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

if either of the following occurs:

$$\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \to \frac{0}{0} \quad \textbf{or} \quad \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \to \frac{\pm\infty}{\pm\infty}$$

then provided the limit exists:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{d}{dx}[f(x)]}{\frac{d}{dx}[g(x)]} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

NOTE: The use of L'Hôpital's Rule is only appropriate for limits in the form of $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

Example 1: Evaluate the limit: $\lim_{x\to 3} \frac{x^2-9}{x-3}$

For this expression, the numerator and denominator both have a limit of 0. So, this limit is the indeterminate form of $\frac{0}{0}$. From previous work with limits, we know that we can try to factor and cancel a common factor to evaluate this limit. But, now we also know that we can utilize L'Hôpital's Rule. We'll do both here to illustrate that either are valid methods for computing this limit.

Factoring:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$$

Applying L'Hôpital's Rule:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} \overset{L}{=} \lim_{x\to 3} \frac{\frac{d}{dx}[x^2-9]}{\frac{d}{dx}[x-3]} = \lim_{x\to 3} \frac{2x}{1} = \lim_{x\to 3} 2x = 2\cdot 3 = 6$$

NOTE: When using L'Hôpital's Rule, you should indicate each time it is used and the type of indeterminate form.

Example 2: Evaluate $\lim_{x\to 0} \frac{e^x-1}{3x^2+x}$

$$\lim_{x\to 0} \frac{e^x-1}{3x^2+x} \to \frac{0}{0}, I.F.$$

So,

$$\begin{aligned} \lim_{x\to 0} \frac{e^x-1}{3x^2+x} &\overset{L}{=} \lim_{x\to 0} \frac{e^x}{6x+1} \leftarrow \textit{now try direct substitution} \\ &= \frac{1}{0+1} \\ &= 1 \end{aligned}$$

Example 3: Evaluate $\lim_{x\to\infty} \frac{4x^3-3}{\ln(x)+5}$

$$\lim_{x\to\infty} \frac{4x^3-3}{\ln(x)+5} \to \frac{\infty}{\infty}, I.F.$$

So,

$$\begin{aligned} \lim_{x\to\infty} \frac{4x^3-3}{\ln(x)+5} &\overset{L}{=} \lim_{x\to\infty} \frac{12x^2}{\frac{1}{x}} \leftarrow \textit{simplify expression} \\ &= \lim_{x\to\infty} 12x^3 \\ &= \infty \end{aligned}$$

Example 4: Evaluate $\lim_{x \to 2} \frac{2^x - 4}{x - 2}$

Example 5: Evaluate $\lim_{x \to 0} \frac{\sin(4x)}{\sin(6x)}$

Example 6: Evaluate $\lim_{x \to -\infty} \frac{x^2 - 3}{2 - x^3}$

Example 7: Evaluate $\lim_{x \to \pi} \frac{\cos(x) - 1}{x}$

Product/Difference Indeterminate Forms

For limits that lead to the indeterminate forms of $\pm\infty \cdot 0$ or $\infty - \infty$, we can use algebra to rewrite the expression we are taking the limit of in order to get it into a form where L'Hôpital's Rule is appropriate.

Example 8: Evaluate $\lim\limits_{x\to\infty} x^2 e^{-x}$

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} x^2 \cdot \lim_{x\to\infty} e^{-x} \to \infty \cdot 0, I.F.$$

So, let's try rewriting the expression to get an appropriate form for L'Hôpital's Rule:

$$\begin{aligned} \lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty}\left(x^2 \cdot \frac{1}{e^x}\right) &= \lim_{x\to\infty} \frac{x^2}{e^x} \to \frac{\infty}{\infty}, I.F. \\ &\overset{L}{=} \lim_{x\to\infty} \frac{2x}{e^x} \to \frac{\infty}{\infty}, I.F. \\ &\overset{L}{=} \lim_{x\to\infty} \frac{2}{e^x} \\ &= 0 \end{aligned}$$

Example 9: Evaluate $\lim\limits_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0^+} \frac{1}{x} - \lim_{x\to 0^+} \frac{1}{e^x - 1} \to \infty - \infty, I.F.$$

So, let's try rewriting the expression:

$$\begin{aligned} \lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) &= \lim_{x\to 0^+}\left(\frac{e^x - 1}{x(e^x - 1)} - \frac{x}{x(e^x - 1)}\right) \leftarrow \textit{common denominator} \\ &= \lim_{x\to 0^+} \frac{e^x - 1 - x}{x(e^x - 1)} \to \frac{0}{0}, I.F. \\ &\overset{L}{=} \lim_{x\to 0^+} \frac{e^x - 0 - 1}{1 \cdot (e^x - 1) + (e^x - 0) \cdot x} \leftarrow \textit{Apply LH Rule; use product rule for denominator} \\ &= \lim_{x\to 0^+} \frac{e^x - 1}{e^x - 1 + xe^x} \to \frac{0}{0}, I.F. \\ &\overset{L}{=} \lim_{x\to 0^+} \frac{e^x - 0}{e^x - 0 + 1 \cdot e^x + e^x \cdot x} \leftarrow \textit{Apply LH Rule again} \\ &= \lim_{x\to 0^+} \frac{e^x}{e^x + e^x + xe^x} \\ &= \frac{1}{1 + 1 + 0} \\ &= \frac{1}{2} \end{aligned}$$

Example 10: Evaluate $\lim_{x \to \frac{\pi}{2}^-} \cos(x) \sec(3x)$

Example 11: Evaluate $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$

Power Indeterminate Forms

The limit $\lim_{x\to a} f(x)^{g(x)}$ can lead to the indeterminate forms of 0^0, ∞^0 or 1^∞. We treat each of these three cases by using properties of the natural logarithm and natural exponential functions.

If:

$$\lim_{x\to a} f(x)^{g(x)} \to 0^0 \quad \textbf{\textit{or}} \quad \lim_{x\to a} f(x)^{g(x)} \to \infty^0 \quad \textbf{\textit{or}} \quad \lim_{x\to a} f(x)^{g(x)} \to 1^\infty$$

Then:

1. Let $y = f(x)^{g(x)}$
2. Take the ln of y and use properties to simplify:

$$\ln y = \ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x))$$

3. Evaluate the limit of $\ln y$:

$$\lim_{x\to a} \ln y = \lim_{x\to a} g(x)\ln(f(x)) = L$$

4. Evaluate the original limit using the above results and properties of ln and e:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = e^{\left(\lim\limits_{x\to a} \ln y\right)} = e^L$$

Example 12: Evaluate $\lim_{x\to 0^+} x^x$

$$\lim_{x\to 0^+} x^x \to 0^0, I.F.$$

We get a power indeterminate form, so we can use the previous steps to help us evaluate this limit.

1. Let $y = f(x)^{g(x)}$.

$$\text{Let } y = x^x$$

2. Take the ln of y and use properties to simplify.

$$\ln y = \ln(x^x) = x\ln x$$

3. Evaluate the limit of $\ln y$.

$$\begin{aligned}
\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} x\ln x \to 0\cdot\infty, I.F. \ (\textit{*Need to rewrite expression}) \\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \to \frac{\infty}{\infty}, I.F. \\
&\overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \leftarrow \textit{simplify expression} \\
&= \lim_{x\to 0^+} (-x) \\
&= 0
\end{aligned}$$

4. Evaluate the original limit using the above results and properties of ln and e.

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} y \\ &= \lim_{x\to 0^+} e^{\ln y} \\ &= e^{\left(\lim_{x\to 0^+} \ln y\right)} \leftarrow \textit{Recall: we just found } \lim_{x\to 0^+} \ln y = 0 \\ &= e^0 \\ &= 1\end{aligned}$$

Example 13: Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} \to 1^\infty, I.F.$$

Let $y = \left(1+\frac{1}{x}\right)^{3x}$.

So,

$$\ln y = \ln\left(1+\frac{1}{x}\right)^{3x} = 3x\ln\left(1+\frac{1}{x}\right)$$

Now,

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty} 3x\ln\left(1+\frac{1}{x}\right) \to \infty\cdot 0, I.F. \ (\textit{*Need to rewrite expression}) \\ &= \lim_{x\to\infty}\frac{3\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \to \frac{0}{0}, I.F. \\ &\overset{L}{=} \lim_{x\to\infty}\frac{3\cdot\frac{1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} \leftarrow \textit{Apply LH Rule; use chain rule for numerator} \\ &= \lim_{x\to\infty}\frac{3}{1+\frac{1}{x}} \leftarrow \textit{simplify expression} \\ &= \frac{3}{1+0} \\ &= 3\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} &= \lim_{x\to\infty} y \\ &= \lim_{x\to\infty} e^{\ln y} \\ &= e^{\left(\lim_{x\to\infty}\ln y\right)} \leftarrow \textit{Recall: we just found } \lim_{x\to\infty}\ln y = 3 \\ &= e^3\end{aligned}$$

Example 14: Evaluate $\lim_{x \to 0^+} (1 - 2x)^{1/x}$

Example 15: Evaluate $\lim_{x \to \infty} x^{(5/x^2)}$